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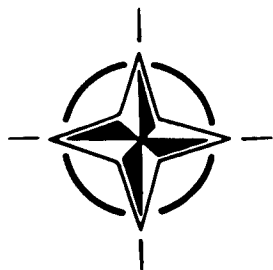
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## **Combat Aircraft Noise**

(Le Bruit Généré par les Avions de Combat)



**NORTH ATLANTIC TREATY ORGANIZATION**

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# Combustion Noise and Combustion Instabilities in Propulsion Systems

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## Abstract

This paper is concerned with some aspects of non-linear behavior of unsteady motions in combustion chambers. The emphasis is on conditions under which organized oscillations having discrete frequencies may exist in the presence of random motions. In order to treat the two types of motions together, and particularly to investigate coupling between noise and combustion instabilities, the unsteady field is represented as a synthesis of acoustic modes having time-varying amplitudes. Each of the amplitudes are written as the sum of two parts, one associated with the random field and the remainder representing the organized oscillations. After spatial averaging, the general problem is reduced to solution of a set of second-order ordinary differential equations whose structure depends on the sorts of nonlinear processes accounted for. This formulation accommodates any physical process; in particular, terms are included to represent noise sources, although only limited modeling is discussed here. Our results suggest that random sources of noise have only small effects on combustion instabilities and seem not to be a cause of unstable motions. However, the coupling between the two sorts of unsteady motions may be important as an essential process in a proposed scheme for noise control.

It is now a familiar observation that many nonlinear deterministic systems are capable of exhibiting apparently random motions called 'chaos.' This is a particularly interesting possibility for systems which also execute non-deterministic random motions. In combustion chambers, a nonlinear deterministic system (acoustical motions) exists in the presence of noise produced by flow separation, turbulent motions, and energy released by combustion processes. The last part of the paper is directed to the matter of discovering whether or not chaotic motions exist in combustion systems. Analysis has not progressed sufficiently far to answer the question. We report here recent results of processing data taken in one combustor to determine the

dimensions of any attractors in the motions. No evidence has been found for chaos in the strict sense, but the method seems to be an important means of investigating the nonlinear behavior of combustion systems.

## 1. Introduction

Jet aircraft in flight carry two types of noise sources: external and internal. External sources include boundary layers, separation regions on the airframe, and especially jet noise associated with the exhaust flow. We are concerned here with some aspects of internal noise generated in combustion chambers.

Following a period of considerable research on the subject of combustion noise during the 1970s, relatively little attention seems to have been directed to theoretical aspects of the subject. A good review of much of the work has been given by Strahle (1978). Both experimental and theoretical results were obtained, but it seems a fair statement that the theory of combustion noise has not progressed as far as the theories of external jet noise. The works by Chiu and Summerfield (1973) and by Chiu, Plett, and Summerfield (1975) seem to be the most complete treatments of combustion noise in chambers or ducts, in which the unsteady motions comprise both random fluctuations and coherent oscillations. More recent experimental work has been reported (Poinot *et al.* 1986; Hegde, Reuter, and Zinn 1988) with some analysis of data, but no advances in theory. The general problem of internal noise shares common features with that of external noise, but the differences are sufficiently great that probably a theory of internal noise should take a rather different form.

In the first part of this paper we construct an approximate analysis having such a form as to be applicable only to a restricted class of internal problems. It is important to appreciate, however, that because sources of noise in a flow always are associated with unsteady motions of the fluid

medium, there must be intrinsic similarities between external and internal theories. Hence as a part of the formulation here, we discuss some of those similarities, helping to clarify where the internal and external theories diverge.

The approximate analysis described here is an extension of a formulation used for many years to study combustion instabilities, based on a form of Galerkin's method (Zinn and Powell 1971 and Culick 1971, 1975, 1990 for example). As described in Section 2, the main idea is to represent a general unsteady motion in a chamber as a synthesis of orthogonal acoustic modes of the chamber, computed for the same geometry but with no combustion or mean flow.

Spatial averaging produces a system of ordinary, nonlinear second-order equations in time for time-dependent amplitudes. At this point the chief difference from previous work is that the equations contain random or stochastic sources (unspecified in detail) and correspondingly the amplitudes are sums of deterministic and stochastic parts.

In Section 3 we discuss a procedure for splitting the unsteady field as a superposition of the coherent acoustic and random motions. The results display explicitly coupling between the acoustic motions only, between the acoustic and random motions, and interactions involving only the random motions. In this formulation, the last appear as external stochastic excitation of the acoustical motions.

Following well-established methods (e.g. Krylov and Bogoliubov 1947 for deterministic systems and Stratonovich 1963 for stochastic systems) we then average the equations in time to produce a set of first-order equations used as the basis for the calculations discussed in Section 4. Time-averaging brings errors which can be assessed only by comparison of numerical results with solutions to the second-order equations. Recent work by Jahnke and Culick (1991) has done much to clarify the matter for deterministic systems but we have no corresponding results for systems containing stochastic sources. The results given in Section 4 are preliminary, illustrating possible effects of stochastic sources on stable deterministic motions and on stable limit cycles.

That the unsteady motions in a combustion chamber are intrinsically nonlinear (although the nonlinear effects may often be small in some sense) and that nonlinear deterministic systems are capable of executing apparently random behavior, called 'chaos,' suggests that we should seek the possible existence of chaotic behavior in combustion chambers. The question seems first to have been

raised by Kantor (1984) in respect to the processes in a reciprocating engine and later investigated by Keanini, Yu, and Daily (1989) for a dump combustor. The case of a reciprocating engine is different because of the presence of the ignition spark. In any event, those results seem not to have established unambiguously that chaos has been identified in those systems. Difficulties accompanying the methods of processing time-series data tend to obstruct definite conclusions.

In Section 5 we investigate the possibility of chaotic behavior with analysis of data taken at Caltech in the past few years with a dump combustor. Although the results show the existence of an attractor, it has integral dimension 2, i.e. it is a toroidal attractor incapable of chaotic motions. The behavior can be represented by simple models of nonlinear behavior but satisfactory connection with the analysis described in the earlier sections of the paper has not been completed. Our results have been attained only for one combustion system and no generalization can be attributed to our conclusions.

Recent works demonstrating 'control of chaos' both theoretically and experimentally (see Ott *et al.* 1990; Ditto, Rauseo, and Spano 1990) suggest investigating the possibility of exerting corresponding control of a combustion system. At this time, the possibility is thinly founded and speculative. The idea is the following. Chaotic behavior, motion on a strange attractor, consists of motions in many orbits (modes) simultaneously. Controlling chaos consists in exerting an external influence forcing the system to execute chiefly a low-amplitude stable motion. The 'orbits' in a combustion system are the acoustic modes.

If then the true random behavior (the noise so annoying in practice) is coupled (nonlinearly) to the acoustic modes — i.e. the orbits comprising the attractor — then it may be possible to control the noise field by controlling the components (modes or orbits) of the chaotic motion. Of course it is unnecessary to obscure the basic notion with the language of the theory of chaos and dynamical systems theory. However, it is important to understand the connection in order to make use of theoretical methods and results available from recent work in that field. It is also possible that in the case at hand, nonlinear coupling, and hence energy flow, between the noise and acoustic fields may offer the possibility of control even though the acoustical motions do not occur in chaotic form.

Essentially what we suggest is the possibility of reducing the noise field — or perhaps affecting the spectral distribution — by suitable control of the

acoustic modes of the chamber. Nonlinear coupling of the deterministic and random motions is clearly necessary for this scheme to work. We are far from being able to check the validity of this idea in respect to combustion systems. Uncertainty is associated with this particular application, not the general idea, which has been confirmed by results obtained with other systems. Whatever the mechanism, some experimental results (Wilson *et al.* 1991; Gulati and Mani 1990; and Poinso *et al.* 1989) suggest that control of acoustic oscillations can be used to control the random fluctuations as well. Figure 1, taken from Poinso *et al.* 1989 shows an example. Reduction of the amplitude of an acoustic mode was accompanied by a reduction in the broadband background. This probably shows in the first instance that the presence of organized oscillations increased the noise level. However, the results also confirm a phenomena already established in experiments with open flames (e.g., Rapp and Schneider 1972; Beckert and Pfizennaies 1972) that the processes generating noise are measurably influenced by an acoustic field. Partly because the more recent works were intended to control the acoustic oscillations only, the matter of controlling noise in this fashion has not been explored either analytically or experimentally.

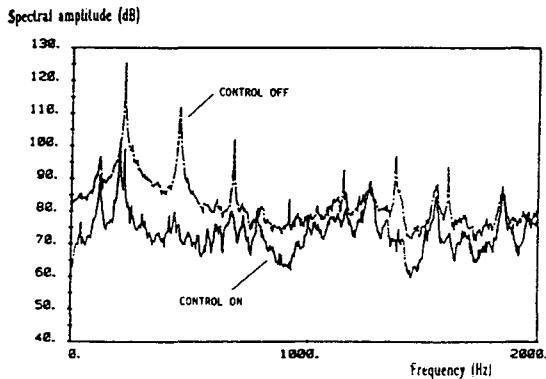


Figure 1.

We must emphasize also that even should control of noise be established as a possibility in principle, the great problem remains of practical implementation. Active control of combustion instabilities, without regard for noise in the system, is a subject of current research (McManus, Poinso, and Candel 1991 have given a recent review of the subject). Promising results have been obtained in laboratory devices using gaseous fuels and oxidizers. Feedback control of an acoustic generator (loudspeaker) or of a secondary supply of fuel has been used successfully to reduce the levels of unstable oscillations. So far as we know, correspondingly successful results have not been reported for liquid-fueled systems.

## 2. Formulation of an Approximate Analysis

The sort of analysis described here has been used for many years to investigate combustion instabilities in solid propellant rockets and liquid-fueled systems, mainly ramjet engines. Hence we need only summarize the formulation here with a view partly to comparing this analysis and the theory of combustion instabilities to the well-known theory of aerodynamic noise initiated by Lighthill (1952, 1953).

In order to accommodate the processes occurring in a combustion chamber, we begin with the complete conservation equations for the gas phase with sources of mass, momentum, and energy:

*Mass*

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = W \quad (2.1)$$

*Momentum*

$$\frac{\partial (\rho \vec{u})}{\partial t} + \nabla \cdot (\rho \vec{u} \vec{u}) = \nabla \cdot \vec{\tau} + \vec{F} \quad (2.2)$$

*Energy*

$$\frac{\partial (\rho e_0)}{\partial t} + \nabla \cdot (\rho \vec{u} e_0) = \nabla \cdot (\vec{\tau} \cdot \vec{u}) - \nabla \cdot \vec{q} + Q \quad (2.3)$$

The heat-flux vector is  $\vec{q}$  and the stress tensor is the sum of the isotropic pressure and the viscous stress tensor  $\vec{\sigma}$ :

$$\vec{\tau} = -p\vec{I} + \vec{\sigma} \quad (2.4)$$

It is necessary to specify the forms of the sources  $W$ ,  $\vec{F}$ , and  $Q$  which, for example, may contain the influences of condensed phases and chemical reactions. Owing to uncertainties in the properties of actual systems, it is entirely satisfactory to represent the medium as a single mass-averaged perfect gas having the equation of state

$$p = R\rho T \quad (2.5)$$

and internal energy  $e$ , such that

$$de = C_v dT \quad (2.6)$$

Subtraction of the scalar product of the velocity and (2.2) from (2.3) leads to the equation for the temperature

$$\rho C_v \left( \frac{\partial T}{\partial t} + \vec{u} \cdot \nabla T \right) = -p \nabla \cdot \vec{u} + \Phi - \nabla \cdot \vec{q} + Q - (e_0 - u^2)W - \vec{u} \cdot \vec{F} \quad (2.7)$$

where the dissipation function is

$$\Phi = \vec{\sigma} \cdot \nabla \cdot \vec{u} \quad (2.8)$$

Because the primary source of data for motions in a combustion chamber is measurement of the time-dependent pressure, there has long been strong motivation to build theoretical frameworks around the pressure as the leading dependent variable. Combination of (2.1), (2.5), and (2.8) gives the energy equation written for the pressure

$$\begin{aligned} \frac{\partial p}{\partial t} + \vec{u} \cdot \nabla p \\ = -\gamma p \nabla \cdot \vec{u} + \frac{R}{C_v} \left[ \Phi - \nabla \cdot \vec{q} + Q + \frac{u^2}{2} W - \vec{u} \cdot \vec{F} \right] \end{aligned} \quad (2.9)$$

Finally, the equation for the entropy of the gases follows from its definition

$$T ds = de + pd \left( \frac{1}{\rho} \right) = C_v dT - \left( \frac{p}{\rho} \right) d\rho \quad (2.10)$$

Substitution of (2.7) and (2.1) leads to

$$\rho T \left( \frac{\partial s}{\partial t} + \vec{u} \cdot \nabla s \right) = \Phi - \nabla \cdot \vec{q} + Q - (\gamma e - u^2) W - \vec{u} \cdot \vec{F} \quad (2.11)$$

As a matter of convenience to simplify writing, we re-write the basic equations (2.1), (2.2), (2.7), (2.9), and (2.11) in the forms

$$\frac{D\rho}{Dt} = \mathcal{W} \quad (2.12)$$

$$\rho \frac{D\vec{u}}{Dt} = -\nabla p + \vec{F} \quad (2.13)$$

$$\rho C_v \frac{DT}{Dt} = -p \nabla \cdot \vec{u} + Q \quad (2.14)$$

$$\frac{Dp}{Dt} = -\gamma p \nabla \cdot \vec{u} + \mathcal{P} \quad (2.15)$$

$$\rho \frac{Ds}{Dt} = \frac{1}{T} \mathcal{S} \quad (2.16)$$

where  $D/Dt = \partial/\partial t + \vec{u} \cdot \nabla$  is the usual convective derivative, and

$$\mathcal{W} = -\rho \nabla \cdot \vec{u} + W$$

$$\vec{F} = \vec{F} + \nabla \cdot \vec{\sigma} - \vec{u} W$$

$$Q = \Phi - \nabla \cdot \vec{q} + Q - (e_0 - u^2) W - \vec{u} \cdot \vec{F}$$

$$\mathcal{P} = \frac{R}{C_v} Q + e W$$

$$S = Q - (\gamma - 1) e W = \frac{C_v}{R} [P - a^2 W] \quad (2.17) a - e$$

These equations, (2.12) – (2.16), are the basis for the theories of both combustion instabilities and noise in

combustion systems, but in this form they are much too general. No purpose is served here by tracing detailed construction of the simplified equations actually used in existing works, but it is important to appreciate the gist of the matter in order to understand both the differences and similarities of the two kinds of theories now summarized in turn.

## 2.1 Combustion Instabilities

Extensive observations obtained during several decades for many systems of all types confirm that combustion instabilities are waves in the combustion product gases excited and sustained primarily by the energy released in chemical reactions. The frequencies of oscillations usually are quite close to those estimated with classical results applied to the same geometry, a chamber enclosed by rigid walls, with the speed of sound equal to that for the high temperature products. Hence, it is reasonable to treat all combustion and flow processes as perturbations of the classical problem. Some adjustments must be made in special circumstances but that view is adequate for the purposes here.

Then the dependent variables are written as sums of mean and fluctuating values,  $p = \bar{p} + p'$ , etc. with the mean values independent of time, usually (but not always) a good assumption. Two alternative strategies can be followed to obtain nonlinear equations for the fluctuations: either a nonlinear equation for the pressure can be formed from the primitive equations (2.12) – (2.16), followed by substitution of the assumed forms for the dependent variables, or those forms can be substituted in (2.12) – (2.16) and then the nonlinear wave equation for the pressure fluctuation is constructed. For comparison with later remarks on the theories of noise, we first form the wave equation.

Write (2.15) in the form

$$\begin{aligned} \frac{\partial p}{\partial t} + a_r^2 \nabla \cdot (\rho \vec{u}) \\ = (a_r^2 - a^2) \nabla \cdot (\rho \vec{u}) - (\vec{u} \cdot \nabla p - a^2 \vec{u} \cdot \nabla \rho) + \mathcal{P} \end{aligned}$$

where  $a_r^2 = \gamma p_r / \rho_r$  is a reference speed of sound, properly taken as the average value in the chamber, and  $a^2 = \gamma p / \rho$  is the local value. Then differentiate with respect to time and substitute (2.1) for  $\nabla \cdot (\rho \vec{u})$ ; some rearrangement leads to

$$\begin{aligned} \frac{\partial^2 p}{\partial t^2} - a_r^2 \nabla^2 p = a_r^2 \nabla \cdot (\nabla \cdot \vec{T} - \vec{F}) \\ + \frac{\partial}{\partial t} [(a_r^2 - a^2) \nabla \cdot (\rho \vec{u}) - (\vec{u} \cdot \nabla p - a^2 \vec{u} \cdot \nabla \rho) + \mathcal{P}] \end{aligned} \quad (2.18)$$

where

$$\vec{T} = \rho \vec{u} \vec{u} - \vec{\sigma} \quad (2.19)$$

We note for later discussion that equation (2.18) contains the substance of the principles of conservation of mass, momentum, and energy as well as the equation of state for a perfect gas. The boundary condition for the pressure is formed by taking the scalar product of the outward normal vector  $\hat{n}$  at the boundary surface with the momentum equation (2.13):

$$\hat{n} \cdot \nabla p = -\rho \frac{\partial \vec{u}}{\partial t} \cdot \hat{n} - (\rho \vec{u} \cdot \nabla \vec{u}) \cdot \hat{n} + \vec{F} \cdot \hat{n} \quad (2.20)$$

Rather than use (2.18) to construct the wave equation for the pressure fluctuation, we now follow the second strategy noted above, beginning with the equations obtained from (2.1) – (2.3) written to second order in the fluctuations:

$$\begin{aligned} \frac{\partial \rho'}{\partial t} + \bar{\rho} \nabla \cdot \vec{u}' \\ = \rho' \nabla \cdot \vec{u} - \vec{u} \cdot \nabla \rho' - \nabla \cdot (\rho' \vec{u}') + W' \end{aligned} \quad (2.21)$$

$$\begin{aligned} \bar{\rho} \frac{\partial \vec{u}'}{\partial t} + \nabla p' \\ = -\bar{\rho} (\vec{u} \cdot \nabla \vec{u}' + \vec{u}' \cdot \nabla \vec{u}) \\ - \bar{\rho} \vec{u}' \cdot \nabla \vec{u}' - \rho' \frac{\partial \vec{u}'}{\partial t} + \vec{F}' \end{aligned} \quad (2.22)$$

$$\begin{aligned} \frac{\partial p'}{\partial t} + \gamma \bar{p} \nabla \cdot \vec{u}' \\ = -\vec{u} \cdot \nabla p' - \gamma p' \nabla \cdot \vec{u} \\ - (\vec{u}' \cdot \nabla p' + \gamma p' \nabla \cdot \vec{u}') + \mathcal{P}' \end{aligned} \quad (2.23)$$

These equations have formed the basis for investigating nonlinear combustion instabilities to second order in the gasdynamic fluctuations and first order in the mean flow field. Thus the fluctuations are assumed to represent *only* the acoustic disturbances. The justification for this assumption is the principle most thoroughly discussed by Chu and Kovaszny (1958). In the limit of small amplitude fluctuations, any disturbance can be treated as a synthesis of three independent modes of propagation: acoustic, vortical, and entropic. Hence in the limit of linear behavior, we can study acoustical problems (in particular the stability of waves in a combustion chamber) without regard for other unsteady motions, including turbulent fluctuations. Finite amplitude motions involve nonlinear interactions that mix the modes and the principle of superposition naturally fails.

Chu and Kovaszny estimated the relative magnitude of the various nonlinear interactions coupling the three modes of propagation. A result particularly relevant here is that of the possible sources

of acoustic waves, they found that nonlinear interactions among vorticity disturbances constitute a dominant source of acoustic waves. That conclusion had been reached several years earlier by Lighthill (1952) in his theory of the generation of aerodynamic noise. In combustion chambers, generation of pressure waves by unsteady burning seems to be the dominant source.

Differentiation of (2.23) with respect to time and substitution of (2.22) for  $\nabla \cdot (\partial \vec{u}' / \partial t)$  leads to the wave equation used in the remainder of this discussion:

$$\nabla^2 p' - \frac{1}{\bar{a}^2} \frac{\partial^2 p'}{\partial t^2} = h \quad (2.24)$$

where  $\bar{a}^2 = \gamma \bar{p} / \bar{\rho}$  is the average speed of sound assumed to be uniform in the chamber, and the right-hand side is

$$\begin{aligned} h = & -\bar{\rho} \nabla \cdot (\vec{u} \cdot \nabla \vec{u}' + \vec{u}' \cdot \nabla \vec{u}) \\ & + \frac{1}{\bar{a}^2} \frac{\partial}{\partial t} (\vec{u} \cdot \nabla p' + \gamma p' \nabla \cdot \vec{u}) \\ & - \bar{\rho} \nabla \cdot \left( \vec{u}' \cdot \nabla \vec{u}' + \frac{\rho'}{\bar{\rho}} \frac{\partial \vec{u}'}{\partial t} \right) \\ & + \frac{1}{\bar{a}^2} \frac{\partial}{\partial t} (\vec{u}' \cdot \nabla p' + \gamma p' \nabla \cdot \vec{u}') \\ & + \nabla \cdot \vec{F}' - \frac{1}{\bar{a}^2} \frac{\partial \mathcal{P}'}{\partial t} \end{aligned} \quad (2.25)$$

The associated boundary condition, formed as the scalar product of  $\hat{n}$  and (2.22) is

$$\hat{n} \cdot \nabla p' = -f \quad (2.26)$$

with

$$\begin{aligned} f = & \bar{\rho} \frac{\partial \vec{u}'}{\partial t} \cdot \hat{n} + \bar{\rho} (\vec{u} \cdot \nabla \vec{u}' + \vec{u}' \cdot \nabla \vec{u}) \cdot \hat{n} \\ & + \bar{\rho} (\vec{u}' \cdot \nabla \vec{u}') \cdot \hat{n} + \rho' \frac{\partial \vec{u}'}{\partial t} \cdot \hat{n} - \vec{F}' \cdot \hat{n} \end{aligned} \quad (2.27)$$

In accord with earlier remarks, we have in mind problems in which the primed quantities represent the sum of acoustic and non-acoustic fluctuations, the latter taken here to be ‘noise’ associated chiefly with vortical fluctuations. We simply ignore the existence of entropic disturbances (or ‘hot spots’ moving with the local velocity of the medium) justified partly by the estimate provided by Chu and Kovaszny that interactions between the acoustic and vortical fields with entropic disturbances are relatively weak within the volume of the chamber. Interactions of entropic waves with a boundary — particularly a choked nozzle — can produce sound waves, although this seems also a lesser effect (Menon and Jou 1990). Hence whereas in previous analyses of combustion instabilities the fluctuations represented the acoustic field, now we must treat the acoustic and vortical fields

together. All fluctuating quantities will therefore be written as sums of acoustic properties, denoted  $(\ )^a$  and properties associated mainly with the vortical motions, identified by superscript  $(\ )^v$ , for example

$$p' = p^a + \tilde{p}, \quad \vec{u}' = \vec{u}^a + \vec{\tilde{u}}, \quad \rho' = \rho^a + \tilde{\rho}, \dots \text{etc.} \quad (2.28)$$

In this way we shall be able to accommodate combustion instabilities and combustion noise in the same framework, founded on applying a method of spatial averaging to the system of equations derived above.

The idea is that any unsteady motion in the chamber can be synthesized of an appropriate set of modes  $\psi_j(\vec{r})$  of the chamber with time-varying amplitudes  $\eta_j(t)$ ; the amplitudes are defined by expressing the pressure fluctuation in the form

$$p'(\vec{r}, t) = \bar{p} \sum_{j=1}^{\infty} \eta_j(t) \psi_j(\vec{r}) \quad (2.29)$$

Thus when we write the variables in the form (2.28), it is the amplitudes that will be split into the acoustic and vortical (more generally, 'non-acoustic') parts. The mode shapes  $\psi_j(\vec{r})$  depend on the geometry of the chamber and on the boundary conditions; if the exhaust nozzle is choked, and the Mach number at the entrance is not too large, it is often a good approximation to set the normal gradient of  $\psi_j$  equal to zero. Whatever the case, we assume that an orthogonal set of modes is available and here we assume that the  $\psi_n$  satisfy

$$\begin{aligned} \nabla^2 \psi_n + k_n^2 \psi_n &= 0 \\ \hat{n} \cdot \nabla \psi_n &= 0 \end{aligned} \quad (2.30)a, b$$

where  $k_n$  is the wavenumber of the  $n^{\text{th}}$  mode. Then straightforward computations (Culick 1975, 1988; Culick and Yang 1989) lead to the equations for the amplitudes

$$\frac{d^2 \eta_n}{dt^2} + \omega_n^2 \eta_n = F_n \quad (2.31)$$

where  $\omega_n$  is the frequency of the  $n^{\text{th}}$  mode,  $\omega_n = \bar{a} k_n$ , and

$$F_n = -\frac{\bar{a}^2}{\bar{p} E_n^2} \left\{ \int h \psi_n dV + \oint f \psi_n dS \right\} \quad (2.32)$$

$$E_n^2 = \int \psi_n^2 dV \quad (2.33)$$

Equation (2.31) represents a set of coupled nonlinear equations, one associated with each of the modes  $\psi_n$ . For analysis of combustion instabilities, satisfactory results are often obtained by retaining only a small number of modes (Paparizos and Culick 1989a, Jahnke and Culick 1991), but representation

of a noise field requires that the series (2.29) contain high-order modes as well.

The set (2.31) has been the basis for routine analysis of combustion instabilities in solid propellant rockets. Good results have been obtained for linear stability and nonlinear behavior; some important unresolved questions remain, notably satisfactory explanation of nonlinear stability or 'triggering' (Paparizos and Culick 1991). This formalism has also been used to study ramjet engines; with suitable modeling of the special physical processes involved, the approach is applicable to afterburners and liquid rockets.

Before analyzing interactions between a noise field and combustion instabilities, we discuss briefly the relation of this formulation to previous analyses of combustion noise.

## 2.2 Combustion Noise

Much of the theoretical work on combustion noise has been founded on Lighthill's theory of aerodynamic noise, so it is appropriate to begin with a few remarks on that theory. The chief purposes here are to make connections with the formulation described here and to emphasize certain items that arise in our treatment of noise and combustion instabilities within a chamber. Hence the following is but a brief and incomplete summary of a few general aspects of the theory. In particular, we do not deal with the details of the most important part of the subject — the source of noise. Moreover, this is not a critical survey of previous works: ample coverage of the approximations involved in theory, and the extent to which they may be valid, has been given in several review papers cited later.

Lighthill's theory (1952) begins with rearrangement of the conservation equations to give a wave equation for the density. The corresponding equation for the pressure was subsequently used by several writers (Lilley 1973; Doak 1973; Howe 1975). As noted earlier, we prefer pressure as the primary variable for investigating unsteady motions in combustion chambers. The crucial idea introduced by Lighthill that made possible substantial progress has been given the name "acoustic analogy." Instead of treating the nonlinear wave equation, he formed the difference between the equations for the density fluctuations in the actual flow field and those in a gas at rest.

The acoustic analogy for the pressure is expressed formally by combining the equations for conservation of mass and momentum in the following way. Differentiate (2.1) with respect to time, substitute

(2.2) for  $\partial(\rho\vec{u})/\partial t$ , and subtract  $a_r^2\nabla^2\rho$  from both sides to find

$$\begin{aligned} \frac{\partial^2\rho}{\partial t^2} - a_r^2\nabla^2\rho \\ = \nabla \cdot \left[ \nabla \cdot (\rho\vec{u}\vec{u} - \vec{\sigma}) + \nabla \cdot (p - a_r^2\rho) \vec{I} \right] \\ - \left( \nabla \cdot \vec{F} - \frac{\partial W}{\partial t} \right) \end{aligned} \quad (2.34)$$

With no sources of mass and momentum ( $W$  and  $\vec{F}$  set equal to 0) this becomes Lighthill's basic equation. To find the corresponding equations for pressure (Lilley 1973), remove the terms  $a_r^2\nabla^2\rho$  from both sides of (2.34), multiply the equation by  $a_r^2$ , and add  $\partial^2\rho/\partial t^2$ , giving

$$\begin{aligned} \frac{\partial^2 p}{\partial t^2} - a_r^2\nabla^2 p \\ = a_r^2\nabla \cdot [\nabla \cdot (\rho\vec{u}\vec{u} - \vec{\sigma})] \\ + \frac{\partial}{\partial t} \left[ \frac{\partial}{\partial t} (p - a_r^2\rho) + a_r^2W \right] - a_r^2\nabla^2\vec{F} \end{aligned} \quad (2.35)$$

Note that neither (2.34) nor (2.35) involve the energy equation. But the second must be consistent with equation (2.18); the right-hand sides are identical only if the following condition is imposed:

$$\begin{aligned} \frac{\partial}{\partial t} (p - a_r^2\rho) + a_r^2W \\ = (a_r^2 - a^2) \nabla \cdot (\rho\vec{u}) - (\vec{u} \cdot \nabla p - a^2\vec{u} \cdot \nabla \rho) + \mathcal{P} \\ = (a_r^2 - a^2) \rho \nabla \cdot \vec{u} - \vec{u} \cdot \nabla (p - a_r^2\rho) + \mathcal{P} \end{aligned}$$

This can be written in the appealing form

$$\frac{D}{Dt} (p - a_r^2\rho) = (a_r^2 - a^2) \rho \nabla \cdot \vec{u} + \mathcal{P} - a_0^2W \quad (2.36)$$

which is actually an identity following from the equations for conservation of mass (2.12) and energy (2.15). Upon substituting (2.17)d and (2.16) for  $\mathcal{P}$ , we have

$$\frac{D}{Dt} (p - a_r^2\rho) = \frac{R}{C_v} \rho T \frac{Ds}{Dt} + (a^2 - a_r^2) \frac{D\rho}{Dt} \quad (2.37)$$

Thus if the speed of sound is constant, taken to be the reference value, the combination  $p - a_r^2\rho$  for a fluid element changes only if the entropy varies.

Following Lighthill's original works in 1952 and 1953, many calculations exist for aerodynamic sound fields, based on the acoustic analogy expressed by equation (2.34). The idea is that the right-hand side is somehow specified so the equation is an inhomogeneous linear wave equation readily solved by well-known methods. The most common problem treated is the sound field emitted by a jet, motivated especially by the practical need to reduce

the noise produced by jet aircraft. Thus the sound field must be computed in the atmosphere outside the region where the sources reside. The problem comes down entirely to specifying the distribution of sources, the right-hand side of (2.33). In fact, it would be enough to be able to specify the source distribution (of monopoles and dipoles) on a surface enclosing the region containing the true sources, a point of view discussed by Ffowcs-Williams (1974). That property implies an intrinsic ambiguity in the problem: two source distributions that differ by a non-radiating distribution produce the same distribution field outside the region in question. Consequently, measurements of the radiated field cannot be interpreted to give a unique source distribution.

Hence a complete solution to the problem of sound produced, for example, by a jet, can be had only by addressing the very difficult theoretical and experimental matters associated with the source distribution in the volume containing the responsible fluctuations. To make connection with combustion instabilities and combustion noise we need to discuss briefly some aspects of the theoretical work, keeping in mind the noise produced by an exhaust jet as the canonical example.

For isothermal flows, Lighthill showed that the dominant term in the right-hand side of (2.34), or equivalently (2.35), is that involving the velocities; the density can be approximately constant,  $\rho = \bar{\rho}$  and with  $a_r = \bar{a}$ , the governing equation (2.35) for the pressure field is

$$\frac{\partial^2 p}{\partial t^2} - \bar{a}^2\nabla^2 p = \bar{a}^2\nabla \cdot [\nabla \cdot (\bar{\rho}\vec{u}\vec{u})] \quad (2.38)$$

The great difficulty in solving (2.38) arises from the fact that within the source region, where the right-hand side is non-zero, the motions comprise both those causing the acoustic field (in particular, turbulent fluctuations) and the acoustic field itself. But to obtain solutions easily, one needs to know the source terms independently of the acoustic pressure (and hence velocity). Moreover, the mean flow field exerts at least two important influences: the sources are carried with the flow, and a non-uniform flow causes refraction of the acoustic field.

To obtain solutions to (2.35), Lighthill's idea was to replace the true sources by an equivalent source distribution stationary with respect to the ambient field outside their region of activity. ('Solving' equation (2.38) means in this context finding a formula with which one may compute the sound field outside the region containing the sources; see, for example, equation 2.42.) In this first instance, it is clearly convenient simply to ignore convection



and refraction. In principle, as Lighthill noted, if one could specify the equivalent source field accurately, both convection and refraction could be accounted for. In the interests of obtaining results with some ease most investigators have constructed equivalent source distributions without trying to model the effects of convection and refraction: the assumption is not so serious as one might expect. Nevertheless, the problem of accounting for convection and refraction within the acoustic analogy seems not to have been satisfactorily solved.

Perhaps the best known attempts to account for some of the influences of the mean flow are Phillips' (1960) derivation of a convective wave equations (his work was followed by others — see Doak 1973 and Ribner 1981 for summaries) and the work by Mani (1972, 1973), who treated relatively simple cases to display possible effects of convection. Schubert (1972) has examined the problem of refraction using numerical solutions with some success.

It seems fair to conclude that the influences of convection and refraction, while not always small (particularly in supersonic jets), are nonetheless secondary. Hence in low-speed flows, one might suspect that, using Lighthill's acoustic analogy with an equivalent stationary source distribution, the results obtained should be a reasonably good first approximation. That has essentially been the view of those working in the field of combustion noise.

Essentially two theoretical treatments of combustion noise survived the efforts of the 1970s, those of Strahle (1971, 1978, 1985) and of Chiu and Summerfield (1973). The first follows Lighthill's acoustic analogy and the second is based on Phillips' form of the analogy using the convective wave equation. In each case, the transition to a treatment of combustion noise is effected by adding to the equations terms representing heat addition, internal heat transfer, and chemical reactions. Roughly that means in the present context that the starting point is equation (2.18) with no mass and momentum sources  $\vec{F}$  and  $W$  being dropped, giving first the equation Strahle chose as his starting point

$$\begin{aligned} \frac{\partial^2 p}{\partial t^2} - a_r^2 \nabla^2 p \\ = a_r^2 \nabla \cdot \nabla \cdot (\rho \vec{u} \vec{u} - \vec{\sigma}) \\ + \frac{\partial}{\partial t} [(a_r^2 - a^2) \nabla \cdot (\rho \vec{u}) - (\vec{u} \cdot \nabla p - a^2 \vec{u} \cdot \nabla \rho)] \\ + (\gamma - 1) \frac{\partial}{\partial t} (\Phi - \nabla \cdot \vec{q} + Q) \end{aligned} \quad (2.39)$$

Chiu and Summerfield (1973) and Chiu, Plett, and Summerfield (1975) based their analysis on an equation in which, following Phillips (1960), time derivatives are replaced by convective derivatives.

We shall only quote the result written in a form consistent with the notation used here:

$$\begin{aligned} \frac{D^2}{Dt^2} (\ln p) - \nabla \cdot (a^2 \nabla \ln p) \\ = \gamma (\nabla \vec{u}) : (\nabla \vec{u}) + (\gamma - 1) \frac{D}{Dt} \left( \frac{Q}{p} \right) \end{aligned} \quad (2.40)$$

where now

$$Q = Q + \Phi - \nabla \cdot \vec{q} \quad (2.41)$$

For simplicity we shall not display the detailed form of  $Q$  which in the work cited includes diffusion. The point is that the convective derivative is used, and the dependent variable is  $\ln p$ . Those are the main differences between the two formulations.

Once the general problem has been cast in the framework of the acoustics analogy, two tasks remain: modeling the sources and finding approximate solutions for the particular problems at hand. Finding accurate representations for the sources is much the more difficult matter. Computations from first principles are out of the question and the limited experimental results for details of the flow in combustion zones implies that one must resort to assumptions and approximations that cannot always be finally justified.

It appears that all who have studied combustion noise agree that the dominant sources are associated with time-dependent heat release. Large changes in the rate of energy supplied by combustion changes cause substantial fluctuations of the gas density in both time and space. A combustion zone therefore presents a distribution of monopole sources, markedly more effective radiators than the quadrupoles  $\nabla \cdot \nabla \cdot (\rho \vec{u} \vec{u})$  dominant in non-reacting turbulent flows. Hence Chiu and Summerfield retain only the term  $Q = Q$ , equation (2.41), on the right-hand side of (2.40); similarly, Strahle considers only the term  $\partial Q / \partial t$  on the right-hand side of (2.39).

In both approaches, the equations are linearized and the results obtained differ, apart from some details of the modeling, only because of the presence of the convective derivatives in (2.40). Strahle (1978) has given formulas for the pressure fluctuations in the sound field far from a compact open flame. 'Compact' means that the flame is small compared with the dominant wavelengths of sound. His results, equations (32) and (33) of that paper, are

$$p' \sim \frac{1}{|\vec{r}|} \int \frac{1}{a^2} \frac{\partial Q}{\partial t} dV \quad (2.42)$$

$$\begin{aligned} p' \sim \\ \frac{1}{|\vec{r}|} \int \frac{1}{a^2} \left[ \frac{\partial Q}{\partial t} + \vec{u} \cdot \nabla Q' + \left( \vec{u}' - \frac{T'}{T} \vec{u} \right) \cdot \nabla \vec{Q} \right] dV \end{aligned} \quad (2.43)$$

The sources in the integrand are evaluated at the retard time  $t - |\vec{r}|/\bar{a}$  where  $\vec{r}$  is the position of the observation. Strahle (1985) has since argued that better agreement with observation is obtained with his formula (2.41) not containing the influences of convection. The problem of accounting correctly for the influences of the mean flow, specifically as it introduces convective and refractive effects is a deep theoretical matter intimately connected with the formulation of the acoustic analogy. It seems not to have been satisfactorily solved for non-reacting flows and has received little attention in the context of combustion noise. We shall not pursue the issue here.

Much of the previous work in combustion noise has been devoted to problems associated with open flames on hot jets. The situation is therefore quite similar to that for jet noise: the radiating sources are unconfined. In this paper we are concerned only with problems of noise in combustion chambers. Thus the active regions of the flow are located within confining walls. Consequently, the noise field we wish to analyze is also contained by essentially rigid boundaries. Fluctuations at the exit plane constitute sources for noise radiated through the exhaust nozzle and ultimately to the outside world. We shall not treat the problem of emission.

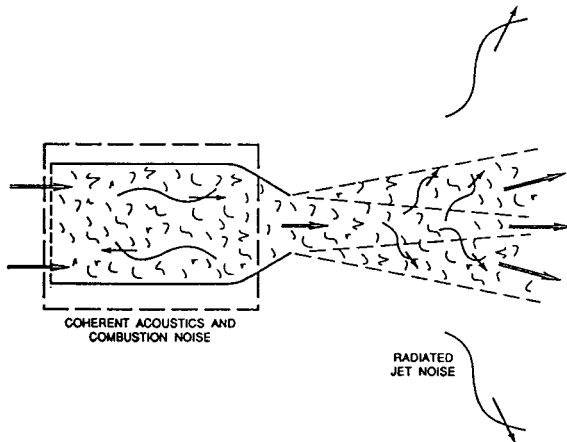


Figure 2.

Figure 2 is a schematic illustration of the matter. The fundamental difference between the problems of confined sources on the one hand and sources freely radiating to an ambient atmosphere on the other has substantial influence on the development of the theory. We shall be able to express the noise field, as well as the acoustic field associated with combustion instabilities, as expansions in natural modes of the chamber. That is why the two classes of phenomena can be treated to good approximation within the same general framework.

Chiu, Plett, and Summerfield (1975) carried out the

first analysis of combustion noise in ducts, including the presence of resonances. Their intent was to explain the spectra of noise emitted when a flame, or more extended combustion processes, are enclosed by confining walls, chiefly for application to gas turbines. The linearized form of (2.39), with only fluctuations of energy release included as sources, becomes

$$\frac{D^2 p'}{Dt^2} - \bar{a}^2 \nabla^2 p' = (\gamma - 1) \frac{DQ'}{Dt} \quad (2.44)$$

Calculations were carried out by expanding the pressure in eigenfunctions of uniform duct with uniform mean flow. The spectral distribution of the heat release was specified, either uniform or as one half of a sinusoid in the frequency range 0 - 1,000 Hz. Then the spatial and time dependence of  $Q'$  were found as an inversion transform in frequency and expansion of the spatial dependence in eigenfunctions of the duct. Equation (2.44) was solved by taking the transform in time, with expansion in eigenfunctions for the spatial dependence of the pressure. Results were then found for the spectral distribution of the pressure.

Figures 3 and 4 show an example of results obtained for the velocity fluctuation at the exit of the duct and its frequency spectrum for a case length/diameter = 8.5. As one should expect, resonance peaks are evident in both figures.

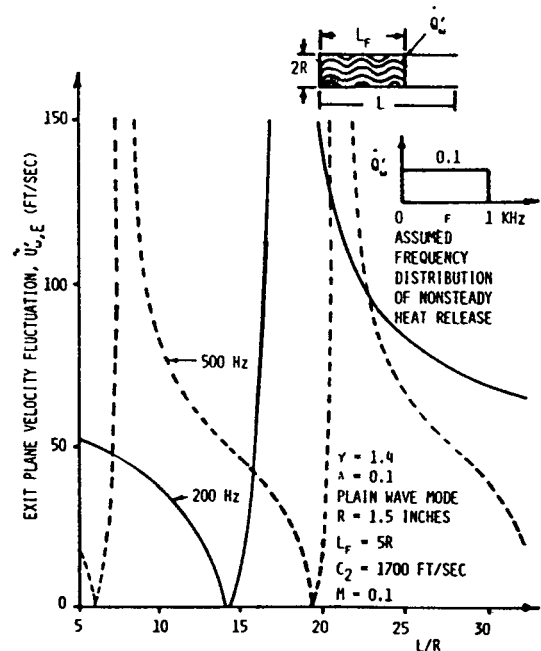


Figure 3.

This method is a straightforward approach to the problem, amounting to computation of the

response of a linear system, the acoustic field in the duct, to a forcing function, the unsteady heat release, consisting of a continuous distribution of sinusoids having spatial dependencies set by the eigenfunctions of the duct. It suffers from at least three serious flaws: the noise sources are independent of the acoustic field, the stochastic nature of the noise field is totally absent, and no nonlinear processes are accounted for. Part of the purpose of the work discussed here is to indicate how these omissions may be corrected.

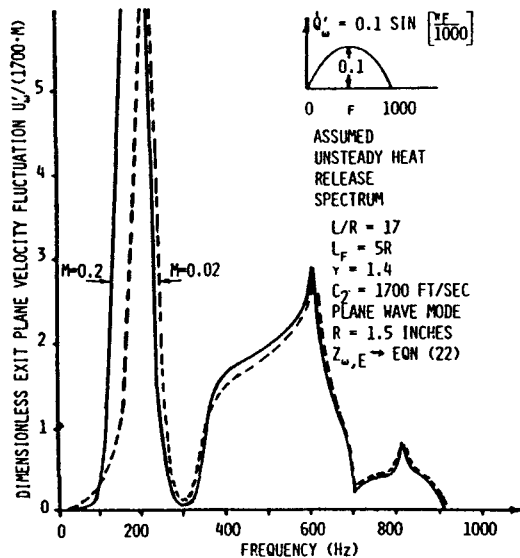


Figure 4.

During the 1980s, interest in combustion noise again grew in connection with work related to oscillators in solid propellant rockets and ramjet engines. Hegde and Strahle (1985) reported their results for the generation of acoustic modes by turbulence in cavities, but no combustion. Their analysis is somewhat closer to actual physical problems for two reasons: they recognize explicitly the different character of irrotational (acoustic) and rotational (vortical) fluctuations; and they account for the spectral content of the vortical or turbulent fluctuations. The analysis naturally contains many approximations; because it is based on a different method of handling the equations of motion, involving definition of the 'Bernoulli enthalpy' (Yates and Sandri 1976), we shall not discuss the calculations. They solve a linear problem of nonreacting flow through a porous lateral boundary and exhausting through a choked exhaust nozzle. Their results show the presence of resonances in satisfactory agreement with observed behavior. The levels of oscillations do not agree well and there is very poor agreement between the

measured and computed spectra for the broadband noise.

Closer to the actual problem of combustion noise is the analysis reported by Hegde, Reuter, and Zinn (1988). For the special problem of a compact flame stabilized on a rod in a duct, they solve the linear problem for the pressure fluctuations, essentially using the linearized form of equation (2.39) with only the term  $(\gamma - 1)\partial Q/\partial t$  on the right-hand side. The principal difference is that they correctly account for the different speeds of sound up- and downstream of the flame. Their most important contribution is their treatment of the statistical properties of the background noise, based on the observed spectrum of the radiation field in turn related to the energy release according to a known method. Their results show quite good agreement between predictions and measurements of the pressure spectrum. That work seems to be the best confirmation of the internal consistency of the formulation described earlier and expressed by equation (2.39), although only linear behavior was treated.

Poinsot *et al.* 1986 have reported experimental work with a dump combustor operated over a sufficiently broad range of conditions that measurements could be taken of the noise field with and without substantial acoustic oscillations. As seems often to be the case for such configurations, the noise level is higher over the entire frequency range when an acoustic oscillation is present. A particularly interesting aspect of that work was the use of optical measurements taken simultaneously to form spatial maps of coherence functions strongly suggestive of the regions in which the sources of noise are located. Information of this sort is clearly useful in modeling sources required for analysis of the phenomena.

These brief remarks are intended only to provide relevant background for the sorts of problems considered in the remainder of the paper. There is much data in the works cited, and others, that we shall not review here. Moreover, we shall not be concerned with the important problem of core noise, under some conditions a strong source of emissions from gas turbines; see the work by Cumpsty and Marble (1977) for the most comprehensive treatment of the subject.

### 3. Splitting the Unsteady Field into Acoustic and Non-Acoustic Fluctuations

The basis for the analysis developed here is combination of the ideas discussed by Chu and Kovasznay (1958) with the approximate analysis developed previously for studying combustion instabilities. As we already noted in Section 2, equa-

tion (2.29), we assume that any unsteady field in a combustion chamber can be synthesized of normal modes of the chamber, with time-varying amplitudes,  $\eta_n(t)$ . After spatial averaging, the problem comes down to solving the system of nonlinear equations (2.30) for the  $\eta_n(t)$ . The right-hand sides will contain nonlinear terms of three sorts representing interactions among the acoustic modes themselves, among the non-acoustic motions, and coupling between the acoustic and non-acoustic fluctuations. Apart from use of spatial averaging, the analysis here differs from that of Chu and Kovaszny (1958) in the explicit consideration of nonlinear processes. Their treatment rested on a linearization of the equations of motion by successive or iterative approximation. Also, because eventually we divide the analysis into the two separate problems of finding the acoustic and non-acoustic amplitudes, we have the possibility of investigating stochastic processes. Therefore, unlike the case for combustion instabilities, in which only a few modes are normally present, here we must expect that a broad spectrum of modes must be present.

First we need to write the formula (2.32) for the  $F_n$  explicitly showing dependence on the fluctuations. We assume that the average thermodynamic properties ( $\bar{p}$ ,  $\bar{\rho}$ ,  $\bar{\tau}$ ) are constant in time and uniform in space. The surface integrals arising from the first, second, and last terms of  $f$ , equation (2.27), can be combined with corresponding volume integrals of  $h$  to give

$$\begin{aligned}
 -\frac{\bar{p}E_n^2}{\bar{a}^2}F_n = & \bar{\rho} \int (\vec{u} \cdot \nabla \vec{u}' + \vec{u}' \cdot \nabla \vec{u}) \cdot \nabla \psi_n dV \\
 & + \frac{1}{\bar{a}^2} \frac{\partial}{\partial t} \int (\vec{u} \cdot \nabla p' + \gamma p' \nabla \cdot \vec{u}) \psi_n dV \\
 & + \bar{\rho} \int \left( \vec{u}' \cdot \nabla \vec{u}' + \frac{\rho'}{\bar{\rho}} \frac{\partial \vec{u}'}{\partial t} \right) \cdot \nabla \psi_n dV \\
 & + \frac{1}{\bar{a}^2} \frac{\partial}{\partial t} \int (\vec{u}' \cdot \nabla p' + \gamma p' \nabla \cdot \vec{u}') \psi_n dV \\
 & + \bar{\rho} \iint \frac{\partial \vec{u}'}{\partial t} \cdot \hat{n} \psi_n dS \\
 & + \int \left( \vec{F}' \cdot \nabla \psi_n - \frac{\partial \mathcal{P}'}{\partial t} \psi_n \right) dV \quad (3.1)
 \end{aligned}$$

Now use a vector identity to rewrite the integrand in the first integral

$$\begin{aligned}
 \vec{u} \cdot \nabla \vec{u}' + \vec{u}' \cdot \nabla \vec{u} \\
 = \nabla \cdot (\vec{u} \vec{u}') - \vec{u}' \times \nabla \times \vec{u} - \vec{u} \times \nabla \times \vec{u}'
 \end{aligned}$$

A second identity gives

$$\begin{aligned}
 \nabla \cdot (\vec{u} \vec{u}') \cdot \nabla \psi_n &= \nabla \cdot (\vec{u} \cdot \vec{u}' \nabla \psi_n) - (\vec{u} \cdot \vec{u}') \nabla^2 \psi_n \\
 &= \nabla \cdot (\vec{u} \cdot \vec{u}' \nabla \psi_n) + k_n^2 \vec{u} \cdot \vec{u}' \psi_n
 \end{aligned}$$

Then because we assume  $\hat{n} \cdot \nabla \psi_n = 0$ , we find

$$\begin{aligned}
 \int (\vec{u} \cdot \nabla \vec{u}' + \vec{u}' \cdot \nabla \vec{u}) \cdot \nabla \psi_n dV = \\
 k_n^2 \int (\vec{u} \cdot \vec{u}') \psi_n dV \\
 - \int (\vec{u} \cdot \nabla \times \vec{u}' + \vec{u}' \times \nabla \times \vec{u}) \cdot \nabla \psi_n dV
 \end{aligned}$$

and (3.1) becomes

$$\begin{aligned}
 -\frac{\bar{p}}{\bar{a}^2} E_n^2 F_n = & \bar{\rho} \int (\vec{u} \cdot \vec{u}') \psi_n dV \\
 & - \bar{\rho} \int (\vec{u} \times \nabla \times \vec{u}' + \vec{u}' \times \nabla \times \vec{u}) \cdot \nabla \psi_n dV \\
 & + \frac{1}{\bar{a}^2} \frac{\partial}{\partial t} \int (\vec{u} \cdot \nabla p' + \gamma p' \nabla \cdot \vec{u}) \psi_n dV \\
 & + \bar{\rho} \int \left( \vec{u}' \cdot \nabla \vec{u}' + \frac{\rho'}{\bar{\rho}} \frac{\partial \vec{u}'}{\partial t} \right) \cdot \nabla \psi_n dV \\
 & + \frac{1}{\bar{a}^2} \frac{\partial}{\partial t} \int (\vec{u}' \cdot \nabla p' + \gamma p' \nabla \cdot \vec{u}') \psi_n dV \\
 & + \bar{\rho} \iint \frac{\partial \vec{u}'}{\partial t} \cdot \hat{n} \psi_n dS \\
 & + \int \left( \vec{F}' \cdot \nabla \psi_n - \frac{\partial \mathcal{P}'}{\partial t} \psi_n \right) dV \quad (3.2)
 \end{aligned}$$

This formula for  $F_n$  shows explicitly only those processes associated purely with gasdynamics. In principle, with  $\vec{F}'$  and  $\mathcal{P}'$ , we can accommodate any physical process. Because the presence of the mean flow field is included to first order — a fundamental matter for problems of linear stability — the effects of convection and refraction are in some sense included here. However, because emphasis has been placed on standing or traveling acoustic waves, these effects have not been a matter of concern for studies of combustion instabilities, in contrast to the case of aerodynamic noise generation. That is, the effects of non-uniform flow to first order have been accounted for in analysis of combustion instabilities, but without attending to the detailed examination required to distinguish the particular consequences of convection and refraction. Some work in this direction has recently been done numerically (e.g. Baum 1988).

Although we shall not discuss extensively the functions  $\vec{F}'$  and  $\mathcal{P}'$ , they contain significant contributions. As we have discussed in the preceding section, the term  $\partial Q'/\partial t$  in  $\partial \mathcal{P}'/\partial t$  has been established as the dominant source for noise production in flames. Unsteady heat release is unquestionably the main reason for both acoustic instabilities and noise in combustion chambers.

However, because they contain important sources of attenuation and noise production due to fluctuating flow variables, the remaining terms in (3.2) must be retained. The fluctuations  $\tilde{\mathcal{F}}'$  and  $\mathcal{P}'$  are essential for analysis of control, for they contain the only means of actuation, through sources of mass, momentum, and energy.

The next step requires expressing the fluctuations  $p'$ ,  $\tilde{u}'$ , and  $\rho'$  in terms of the time-dependent amplitudes  $\eta_n(t)$ . We assume the splitting suggested by equation (2.28) and write the pressure as the expansion (2.29):

$$\begin{aligned} p' &= p^a + \tilde{p} = \tilde{p} \sum_{j=1}^{\infty} \eta_j(t) \psi_j(\vec{r}) \\ &= \tilde{p} \sum_{j=1}^{\infty} (\eta_j^a + \tilde{\eta}_j) \psi_j(\vec{r}) \end{aligned} \quad (3.3)$$

where the  $\eta_j^a$  are the amplitudes of acoustic oscillations (combustion instabilities). The  $\tilde{\eta}_j$  are the amplitudes of the pressure fluctuations associated with the rest of the motions synthesized as a superposition of normal acoustic modes. To compute the amplitudes as solutions to the system (2.31) we need to devise an approximation to  $F_n$ , i.e. we must deal with  $p'$ ,  $\rho'$ , and  $\tilde{u}'$  in the integrals. Following the tactic successfully used for studying combustion instabilities, we assume that the acoustic part of the motions satisfies the classical acoustic equations,

$$\begin{aligned} \frac{\partial \tilde{u}^a}{\partial t} &= -\frac{1}{\bar{\rho}} \nabla p^a \\ \frac{\partial p^a}{\partial t} &= -\gamma \bar{p} \nabla \cdot \tilde{u}^a \end{aligned} \quad (3.4)a, b$$

Because there is no entropy change associated with acoustic waves,  $\tilde{p} + p^a \sim (\bar{\rho} + \rho^a)^\gamma$  and we set  $\rho^a = p^a / \bar{a}^2$ . Hence the expansions of  $\tilde{u}^a$  and  $\rho^a$  are

$$\begin{aligned} \tilde{u}^a &= \sum_{j=1}^{\infty} \frac{\hat{\eta}_j^a}{\gamma k_j^2} \nabla \psi_j \\ \rho^a &= \frac{\bar{p}}{\bar{a}^2} \sum_{j=1}^{\infty} \eta_j^a \psi_j \end{aligned} \quad (3.5)a, b$$

The formulas (2.29) and (3.5)a satisfy equations (3.4)a, b exactly because the eigenfunctions  $\psi_j$  satisfy (2.30)a and in zeroth order  $\tilde{\eta}_j + \omega_j^2 \eta_j = 0$  with  $\omega_j = \bar{a} k_j$ .

Thus we approximate the acoustical variables in  $F_n$  by their values existing in zeroth order with no mean flow or sources. For the remaining parts,  $\tilde{p}$ ,  $\tilde{\rho}$ , and  $\tilde{n}$ , we appeal to results established by Chu and Kovasznay (1958) who showed that the vortical and entropic modes of propagation carry velocity

changes, but no pressure changes. We shall assume also that the density is constant; hence we *assume* in zeroth approximation

$$\tilde{p} = \tilde{\rho} = 0 \quad ; \quad \tilde{u} \neq 0 \quad (3.6)$$

It is not necessary at this stage to take  $\tilde{\rho} = 0$ ; we do so to eliminate a few terms. However, if  $\tilde{\rho}$  is taken to be non-zero, the problem arises later of relating  $\tilde{\rho}$  to other thermodynamic properties. With these approximations, to second order in the fluctuations (3.2) becomes

$$F_n = F_n^a + \tilde{F}_n + \tilde{F}_n^a \quad (3.7)$$

where

$$\begin{aligned} F_n^a &= -\frac{\gamma}{\bar{\rho} E_n^2} \left[ \bar{\rho} k_n^2 \int (\tilde{\vec{u}} \cdot \tilde{\vec{u}}^a) \psi_n dV \right. \\ &\quad - \bar{\rho} \int (\tilde{\vec{u}}^a \times \nabla \times \tilde{\vec{u}}) \cdot \nabla \psi_n dV \\ &\quad \left. + \frac{1}{\bar{a}^2} \frac{\partial}{\partial t} \int (\tilde{\vec{u}} \cdot \nabla p^a + \gamma p^a \nabla \cdot \tilde{\vec{u}}) \psi_n dV \right] \\ &\quad - \frac{\gamma}{\bar{\rho} E_n^2} \left[ \bar{\rho} \int \left( \tilde{\vec{u}}^a \cdot \nabla \tilde{\vec{u}}^a + \frac{1}{\bar{a}^2} p^a \frac{\partial \tilde{\vec{u}}^a}{\partial t} \right) \cdot \nabla \psi_n dV \right. \\ &\quad \left. + \frac{1}{\bar{a}^2} \frac{\partial}{\partial t} \int (\tilde{\vec{u}}^a \cdot \nabla p^a + \gamma p^a \nabla \cdot \tilde{\vec{u}}^a) \psi_n dV \right] \\ &\quad - \frac{\gamma}{E_n^2} \iint \frac{\partial \tilde{\vec{u}}^a}{\partial t} \cdot \hat{n} dS \\ &\quad - \frac{\gamma}{\bar{\rho} E_n^2} \int \left( \tilde{\mathcal{F}}^a \cdot \nabla \psi_n - \frac{\partial \mathcal{P}^a}{\partial t} \psi_n \right) dV \end{aligned} \quad (3.8)$$

$$\begin{aligned} \tilde{F}_n^a &= -\frac{\gamma}{\bar{\rho} E_n^2} \left[ \bar{\rho} \int \left( \tilde{\vec{u}}^a \cdot \nabla \tilde{\vec{u}} + \tilde{\vec{u}} \cdot \nabla \tilde{\vec{u}}^a + \frac{p^a}{\bar{a}^2} \frac{\partial \tilde{\vec{u}}}{\partial t} \right) \cdot \nabla \psi_n dV \right. \\ &\quad \left. + \frac{1}{\bar{a}^2} \frac{\partial}{\partial t} \int (\tilde{\vec{u}} \cdot \nabla p^a + \gamma p^a \nabla \cdot \tilde{\vec{u}}) \psi_n dV \right] \\ &\quad - \frac{\gamma}{\bar{\rho} E_n^2} \int \left( \tilde{\mathcal{F}}^a \cdot \nabla \psi_n - \frac{\partial \tilde{\mathcal{P}}^a}{\partial t} \psi_n \right) dV \end{aligned} \quad (3.9)$$

$$\begin{aligned} \tilde{F}_n &= -\frac{\gamma}{\bar{\rho} E_n^2} \left[ \bar{\rho} k_n^2 \int (\tilde{\vec{u}} \cdot \tilde{\vec{u}}) \psi_n dV \right. \\ &\quad \left. - \bar{\rho} \int (\tilde{\vec{u}} \times \nabla \times \tilde{\vec{u}} + \tilde{\vec{u}} \times \nabla \times \tilde{\vec{u}}) \cdot \nabla \psi_n dV \right] \\ &\quad - \frac{\gamma}{E_n^2} \int (\tilde{\vec{u}} \cdot \nabla \tilde{\vec{u}}) \cdot \nabla \psi_n dV \\ &\quad - \frac{\gamma}{E_n^2} \iint \frac{\partial \tilde{\vec{u}}}{\partial t} \cdot \hat{n} \psi_n dS \\ &\quad - \frac{\gamma}{\bar{\rho} E_n^2} \int \left( \tilde{\mathcal{F}} \cdot \nabla \psi_n - \frac{\partial \tilde{\mathcal{P}}}{\partial t} \psi_n \right) dV \end{aligned} \quad (3.10)$$

With the preceding definitions, the system (2.31) becomes

$$\frac{d^2\eta_n}{dt^2} + \omega_n^2\eta_n = F_n^a + \tilde{F}_n^a + \tilde{F}_n \quad (3.11)$$

The forcing functions  $F_n^a$  and  $\tilde{F}_n$  contain the stochastic properties of the random field.

Previous work (Culick 1975, 1990) has shown that if only the gasdynamic nonlinear processes are accounted for to second order,  $F_n^a$  has the general form

$$F_n^a = \sum_{i=1}^{\infty} [D_{ni}\eta_i^a + E_{ni}\eta_i^a] + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} [A_{nij}\eta_i^a\eta_j^a + B_{nij}\dot{\eta}_i^a\dot{\eta}_j^a] \quad (3.12)$$

where the coefficients  $D_{ni}, \dots$  are constants, depending on the linear processes (including those arising from  $\tilde{F}^a$  and  $\mathcal{P}^a$ ) and on the geometry of the chamber. If other nonlinear processes are accounted for, accommodated by the source functions  $\tilde{F}^a$  and  $\mathcal{P}^a$  (e.g. nonlinear interactions between combustion and the acoustic field), additional terms will appear, not necessarily having the form shown in (3.12).

According to its definition (3.9), the force  $\tilde{F}^a$  contains terms linear in the acoustic amplitudes and in the random variables  $\tilde{u}$  and  $\partial\tilde{u}/\partial t$  plus terms arising from  $\tilde{F}^a$  and  $\tilde{\mathcal{P}}^a$ . If we ignore the possibility of higher order nonlinearities in those source terms, we may assume that  $\tilde{F}_n^a$  has the general form

$$\tilde{F}^a = \sum_{i=1}^{\infty} [\xi_i\eta_i^a + \xi_i^v\dot{\eta}_i^a] \quad (3.13)$$

where the  $\xi_i, \xi_i^v$  represent (stochastic) parametric excitation of the acoustic modes.

Finally,  $\tilde{F}_n$  depends only on the stochastic fluctuations, spatially averaged over the node shapes. To make the notation more consistent, we replace  $\tilde{F}_n$  by the symbol  $\Xi_n$ :

$$\tilde{F}_n \equiv \Xi_n \quad (3.14)$$

With these definitions, we write system (3.11) showing explicitly the dependence on stochastic processes:

$$\frac{d^2\eta_n}{dt^2} + \omega_n^2\eta_n = F_n^a + \sum_{i=1}^{\infty} [\xi_i\eta_i^a + \xi_i^v\dot{\eta}_i^a] + \Xi_n \quad (3.15)$$

Note that the  $\xi_i$  and the  $\xi_i^v$  depend linearly on the  $\tilde{\eta}_i$  and  $\dot{\tilde{\eta}}_i$  while  $\Xi_n$  is quadratic in those variables.

An important fundamental aspect of the systems treated here is the general problem of distinguishing

deterministic and stochastic behavior. The natural splitting of the forces  $F_n$  into the components  $F_n^a$ ,  $\tilde{F}_n^a$ , and  $\tilde{F}_n$  defined by (3.8) – (3.10) is only an initial step. For unsteady motions in combustion chambers, we must pay particular attention to the source functions  $\mathcal{P}^a$ ,  $\tilde{\mathcal{P}}^a$ , and  $\tilde{\mathcal{P}}$ , for they contain representations of energy addition. The acoustic field is purely deterministic, so  $\mathcal{P}^a$  contains only deterministic energy sources. At the other extreme,  $\tilde{\mathcal{P}}$  includes sources of energy addition that are purely stochastic, i.e., depend only on the turbulent motions and are insensitive to the deterministic acoustic field. Interactions between the deterministic and stochastic fields belong to  $\tilde{\mathcal{P}}^a$ . For example, turbulent fluctuations in a reacting shear layer may be sensitive to the local acoustic field, thereby affecting the energy release. A difficult part of the subject is construction of realistic models of these processes.

We have reduced the problem to finding the response of a system of nonlinear oscillators whose amplitudes  $\eta_n = \eta_n^a + \tilde{\eta}_n$  must be determined when the system is subject to stochastic forces. The developments leading to the definitions of  $\xi_i$ ,  $\xi_i^v$ , and  $\Xi_n$  have produced formulas (not given here) showing their explicit dependence on the random fluctuations of the velocity field. Before solutions can be found for the time-dependent amplitudes, those functions should evidently be expressed in terms of the  $\tilde{\eta}_i$  and  $\dot{\tilde{\eta}}_i$ , analogous to the way in which the acoustic contributions in  $F_n$  were treated by using the approximations (3.4)a, b.

That is a large and difficult problem. There are at least four approaches to its solution: (1) construct an approximate representation of the flow variables  $\tilde{u}$ ,  $\tilde{p}$ ,  $\tilde{\rho}$  and the combustion processes, based on observations and guided by theoretical considerations, in the same spirit as we have treated the acoustic field; (2) extract a representation from numerical simulations using methods of computational fluid dynamics; (3) extract a representation by analysis of experimental data; and (4), the crudest approach, assume forms for the various contributions to  $\tilde{F}_n^a$  and  $\tilde{F}_n$  and determine the consequences by solving the system (3.11). None of the first three approaches have been investigated extensively for the class of problems considered here. The works by Chiu, Plett, and Summerfield (1975) and by Hegde, Reuter, and Zinn (1988) fall in the first category, but treat only linear behavior. A few elementary results have been reported by Menon and Jou (1990) following the second approach.

Methods of analyzing experimental data or the results of numerical simulations have become active

subjects of research in the past few years. There is a growing body of literature dealing mainly with non-reacting flows. For example, the method called 'proper orthogonal decomposition,' originated in probability theory, has been applied to turbulent flows (Lumley 1967, Aubrey *et al.* 1988) and other problems of fluid mechanics (Sirovich 1987). That and related methods applied to a wide range of problems were central issues in a recent IUTAM/NATO Workshop (1991). Although those ideas are clearly applicable to the unsteady motions in combustion chambers, no results have been reported.

In order to gain some idea of the possible influences of stochastic sources on the behavior of acoustic modes, we have followed the last of the approaches listed above. We simply assume forms for the random functions of time,  $\xi_i$ ,  $\xi_i^v$ , and  $\Xi_n$ , and treat the system of stochastic differential equations

$$\frac{d^2\eta_n}{dt^2} + \omega_n^2\eta_n = F_n^a + \sum_{i=1}^{\infty} [\xi_i\eta_i + \xi_i^v\dot{\eta}_i] + \Xi_n \quad (3.16)$$

This was the approach taken by Papanizos and Culick (1989)*a* for the simplest case of two longitudinal modes. We discuss that method in Section 4. Note that  $\eta_i^a$ ,  $\dot{\eta}_i^a$  have been replaced by  $\eta_i = \eta_i^a + \tilde{\eta}_i$  and  $\dot{\eta}_i = \dot{\eta}_i^a + \dot{\tilde{\eta}}_i$  on the right-hand side, implying the assumption that the random part of the amplitudes is relatively small.

In fact it appears that we may quite realistically assume that the  $\xi_i$ ,  $\xi_i^v$ , and  $\Xi_i$  do *not* depend on the amplitudes  $\eta_n$  of the pressure fluctuations. This assumption has been made in all treatments of the aerodynamic noise problem. In other words, we assume that the random sources of noise do not themselves depend on the noise they generate. That does not exclude dependence of the noise field on the presence of coherent acoustic oscillations, nor does it imply that control of the noise field is impossible. The reason is that the acoustic and random pressure fluctuations are coupled by nonlinear processes, producing energy transfer between the two forms of motion. Practical matters of achieving successful control will rest partly on the strength of the energy exchange and on the possibility of discovering an effective means of control or 'actuation.' However, it may also be the case that if the random sources of noise are affected by the pressure fluctuations, then more effective means of control might be found.

Part of our justification for taking the stochastic sources independent of the pressure field rests on the results discussed by Chu and Kovasznay (1958) who established the independence of the three modes of propagation (acoustic, vortical, and entropic). The argument, however, is weak, ignoring some

basic properties of processes present in combustion systems. For example, heat released in flames is indeed sensitive to pressure. Our assumption here is really made in the interests of obtaining some initial results.

#### 4. Combustion Instabilities with Stochastic Sources

The formulation developed in the preceding section accommodates a wide range of problems. Here we shall describe a few results for what is possibly the simplest case: the influence of stochastic sources on a combustion instability. Observations of instabilities in both laboratory and full-scale combustors normally show the presence of a small number of acoustic modes; the data cannot reveal how many modes must be accounted for in an analysis to explain satisfactorily the nonlinear behavior. In the theory used here, a minimum of two modes are required to produce a limit cycle. Energy is supplied to the acoustical system because one mode is unstable; the motions may then reach a steady state only if a stable second mode is available to dissipate energy, thereby providing the possibility for constant total energy in the acoustical motions. Nonlinear gasdynamical processes cause the flow of energy from the unstable mode to the stable mode.

Pressure records of instabilities rarely show purely harmonic motions. The amplitudes for unfiltered data usually fluctuate about some apparent average value. Those fluctuations are erased if the data are filtered. Otherwise, they will cause broadening of peaks in the spectra and as part of the broadband spectral background. The analysis described in this section is an initial effort to explain this behavior. Only limited results have been obtained.

##### 4.1 The Two-Mode Approximation

For the acoustic system described by (3.16) with the stochastic sources equal to zero, and only gasdynamical nonlinear processes accounted for, exact solutions can be found for the time-averaged equations for two (and in some cases three) modes. Most of the known results have been reviewed by Culick (1990). In particular, the conditions for existence and stability of limit cycles can be written explicitly. Although the restriction to two modes carries limitations, serious under some conditions (Jahnke and Culick 1991), this representation is surprisingly accurate under wide realistic circumstances. To maintain simplicity, we therefore consider here only the two-mode approximation.

Moreover, we treat the case of longitudinal modes for which the frequencies are integral multiples of

the fundamental,  $\omega_n = n\omega_1$ . With all linear processes and only the gasdynamic nonlinearities included, the first two of equations (3.16) are

$$\begin{aligned}\ddot{\eta}_1 + \omega_1^2 \eta_1 &= 2(\alpha_1 \dot{\eta}_1 + \theta_1 \omega_1 \eta_1) \\ &\quad - (F_{11} \dot{\eta}_1 \dot{\eta}_2 + F_{12} \eta_1 \eta_2) + (\xi_1 \eta_1^a + \xi_1^v \dot{\eta}_1^a + \Xi_1) \\ \ddot{\eta}_2 + \omega_2^2 \eta_2 &= 2(\alpha_2 \dot{\eta}_2 + \theta_2 \omega_2 \eta_2) \\ &\quad - (F_{21} \dot{\eta}_1^2 + F_{22} \eta_1^2) + (\xi_2 \eta_2^a + \xi_2^v \dot{\eta}_2^a + \Xi_2)\end{aligned}\quad (4.1)a, b$$

where  $\omega_2 = 2\omega_1$  and

$$\begin{aligned}F_{11} &= \frac{3-2\gamma}{2\gamma} & F_{12} &= \frac{5(\gamma-1)}{2\gamma} \omega_1^2 \\ F_{21} &= -\frac{\gamma+3}{2\gamma} & F_{22} &= \frac{\gamma-1}{2\gamma} \omega_1^2\end{aligned}\quad (4.2)$$

There are six stochastic functions in equation (4.1), in general mutually independent, to be specified. The functions bring with them a large number of parameters, too many at this stage for a sensible treatment of the problem. We shall therefore analyze a much simpler problem, motivated by the following reasoning based on previous results (Paparizos and Culick 1989b) obtained for the deterministic problem with no stochastic sources.

Combustion instabilities in practice commonly appear as oscillations having slowly varying maximum amplitudes and phases; the amplitudes  $\eta_i$  may then be assumed to have the form

$$\begin{aligned}\eta_i(t) &= r_i(t) \cos(\omega_i t - \phi_i(t)) \\ &= A_i(t) \cos \omega_i t + B_i(t) \sin \omega_i t\end{aligned}\quad (4.3)$$

where  $\delta r_i/r_i$  and  $\delta \phi_i/2\pi$ , the changes over one cycle, are assumed small. The method of time-averaging is effective under these conditions, reducing the two second order equations for  $\eta_1$  and  $\eta_2$  to four first order equations for the  $(r_1, r_2, \phi_1, \phi_2)$ .

For technical reasons not covered here, the following transformation is an effective choice to simplify the structure of the first order equations:

$$\begin{aligned}y_1 &= r_1 \\ y_2 &= r_2 \sin(\phi_2 - 2\phi_1) \\ y_3 &= r_2 \cos(\phi_2 - 2\phi_1)\end{aligned}\quad (4.4)a, b, c$$

The deterministic equations for the  $y_i$  are:

$$\begin{aligned}\frac{dy_1}{dt} &= (\alpha_1 + \beta y_2) y_1 \\ \frac{dy_2}{dt} &= \alpha_2 y_2 + \theta^* y_3 + 2\beta y_3^2 - \beta y_1^2 \\ \frac{dy_3}{dt} &= -\theta^* y_2 + \alpha_2 y_3 - 2\beta y_2 y_3\end{aligned}\quad (4.5)a, b, c$$

where

$$\begin{aligned}\beta &= \frac{\gamma+1}{8\gamma} \omega_1 \\ \theta^* &= \theta_2 - 2\theta_1\end{aligned}\quad (4.6)a, b$$

Equations (4.5) are unchanged if  $y_3$  is replaced by  $y_3[\theta_2 - 2\theta_1]/(\theta_2 - 2\theta_1)$ , showing that no generality is lost by taking  $\theta^*$  positive. Note that the original set of four equations (not given here) are reduced to three, essentially because there is an arbitrary phase available, the reason that the two phase angles appear only in the combination  $\phi_2 - 2\phi_1$  in (4.4)a, b, c.

Fixed points of the system (4.5), defined by  $\dot{y}_i = 0$ , are the state of rest ( $y_i = 0$ ) or limit cycles. It can be shown (Paparizos and Culick 1989b) that a unique limit cycle, whose properties are independent of initial conditions, exists if  $\alpha_1 \alpha_2 < 0$ . As remarked above, if one mode (say the fundamental) is unstable,  $\alpha_1 > 0$ , the other must be stable,  $\alpha_2 < 0$ . In this case, the limit cycle itself is stable if  $2\alpha_1 + \alpha_2 < 0$ . Hence for existence and stability of the limit cycle, the conditions must be satisfied when  $\alpha_1 > 0$  are

$$\begin{aligned}\alpha_2 &< 0 \\ 2\alpha_1 + \alpha_2 &< 0\end{aligned}\quad (4.7)a, b$$

The remainder of this section is concerned with the influences of stochastic sources on the characteristics of this limit cycle.

## 4.2 Stochastic Averaging

Stratonovich (1963, Vol. II, Chapter 4) introduced the method of stochastic averaging, an extension of the method of averaging outlined above for a deterministic system. See also Roberts and Spanos (1986) and Gardiner (1985) for reviews of the method. Here we simply quote the results of applying the method.

With stochastic sources in the second order equations (4.1)a, b, the averaging proceeds in two steps. First the equations are averaged over the period of the acoustic oscillations ( $\tau_n$  for the  $n^{\text{th}}$  mode) to give, eventually, equations (4.5)a, b, c in the variables  $y_i$ ; that is, the oscillatory motions have been removed from the problem which has been reduced to one of finding the slowly varying amplitudes and phases (equivalent to the  $y_i(t)$ ).

The second step is the 'stochastic averaging,' required to give the fluctuating (stochastic) sources in correct form for the averaged (i.e. first order) equations. We assume that the stochastic sources  $\xi_i(t)$ ,  $\xi_i^v(t)$ , and  $\Xi_i(t)$  have zero mean (eventually we assume that they have the form of white noise). However, the terms  $\xi_i \eta_i$  and  $\xi_i^v \dot{\eta}_i$ , representing parametric excitations, have non-zero mean values.



It is convenient to remove the mean values from the stochastic sources and place them with the deterministic part, accounting for the terms  $m_i$  appearing below. By definition,  $m_i$  is the average over one period of the oscillation of the expected value of the term in question.

It is also assumed that the auto-correlation times of the random processes are much smaller than the bandwidths (not the periods) of the oscillations. That means that the oscillations have non-zero decay (or growth) coefficients such that

$$\tau_{\text{corr}} \ll |\alpha_i| \quad (4.8)$$

a condition satisfied by the broad-band processes and acoustical motions expected in practice. In the limit considered here, we approximate the random processes by equivalent white noise.

Finally, in order to reduce the difficulty and complexity of finding solutions to equations (4.1)a, b we make two assumptions based partly on physical grounds:

(i) We replace  $\eta_i^a$  and  $\dot{\eta}_i^a$  in the terms representing parametric excitation by  $\eta_i$  and  $\dot{\eta}_i$ . This implies that the stochastic perturbations of the acoustic modes should be small:  $|\dot{\eta}_i|/|\eta_i^a| \ll 1$ . That happens to be true in the examples treated here, but not in general.

(ii) We ignore all stochastic sources in the equation (4.1)b for the amplitude of the second mode. This assumption produces considerable simplification, but is also motivated by the expectation that if the fundamental mode has a growth rate much larger than the decay rate of the second mode, then the first-order effects of stochastic sources will be primarily on the fundamental mode. Owing to nonlinear coupling between the modes, there will of course be a secondary random excitation of the second mode.

Omitting the lengthy details of stochastic averaging (which in fact begins with the second order equations) we eventually find the equations for the variables  $y_i$ :

$$dy_1 = \left( \alpha_1^* y_1 + \beta y_1 y_2 + \frac{\pi}{2y_1 \omega_1^2} S_3(\omega_1) \right) dt + \sqrt{Q_{11}} dw_1$$

$$dy_2 = (\alpha_2^* y_2 + \theta^* y_3 + 2\beta y_3^2 - \beta y_1^2) dt - 2y_3 \sqrt{Q_0} dw_2$$

$$dy_3 = (\alpha_2^* y_3 - \theta^* y_2 - 2\beta y_2 y_3) dt + 2y_2 \sqrt{Q_0} dw_3 \quad (4.9)a, b, c$$

The  $w_i(t)$  are Gaussian (white noise) processes, the  $dw_i$  being defined in terms of the stochastic integral of a function  $G(t)$  (see Gardiner 1985, p. 83),

$$\int_{t_0}^t G(t') dw_i(t')$$

The  $S_i(\omega)$  are the spectral density functions of the  $w_i$  and the  $Q_{ij}$  are cross-variances of the stochastic sources, computed in the averaging process

$$Q_{11} = \frac{\pi y_1^2}{4} \left( 2S_1(0) + S_1(2\omega_1) + \frac{1}{\omega_1^2} S_2(2\omega_1) \right) + \frac{\pi}{\omega_1^2} S_3(\omega_1)$$

$$Q_0 = \frac{\pi}{4} \left( \frac{2}{\omega_1^2} S_2(0) + S_1(2\omega_1) + \frac{1}{\omega_1^2} S_2(2\omega_1) + \frac{4}{y_1^2 \omega_1^2} S_3(\omega_1) \right)$$

$$Q_{22} = 4y_3^2 Q_0$$

$$Q_{23} = -4y_2 y_3 Q_0$$

$$Q_{33} = 4y_2^2 Q_0$$

$$Q_{12} = Q_{13} = 0$$

(4.10)a - f

The mean values of the sources are

$$m_1 = \frac{\pi}{4} \left[ y_1 S_1(0) + \frac{3}{2} y_1 \left( S_1(2\omega_1) + \frac{1}{\omega_1^2} S_2(2\omega_1) \right) + \frac{2}{\omega_1^2 y_1} S_3(\omega_1) \right]$$

$$m_2 = 2y_2 Q_0$$

$$m_3 = -2y_3 Q_0$$

(4.11)a, b, c

Finally, the growth and decay rates of the two modes are now

$$\alpha_1^* = \alpha_1 + \frac{\pi}{4} \left( S_1(0) + \frac{3}{2} S_1(2\omega_1) + \frac{3}{2\omega_1^2} S_2(2\omega_1) \right)$$

$$\alpha_2^* = \alpha_2 - \pi \left( \frac{1}{\omega_1^2} S_2(0) + \frac{1}{2} S_1(2\omega_1) + \frac{1}{2\omega_1^2} S_2(2\omega_1) + \frac{2}{\omega_1^2 y_1} S_3(\omega_1) \right) \quad (4.12)a, b$$

### 4.3 The Center Manifold for Equations (4.9)

Although equations (4.9)*a, b, c* can be solved numerically in their present form, it is useful to simplify the matter further by taking advantage of some results obtained by Paparizos and Culick (1989*b*). It is relatively straightforward to find the equation for the center manifold, an approximation to the one-dimensional manifold (or locus) of limit cycles extending from the origin  $y_1 = y_2 = 0$ . We expect that a similar result should be a good approximation here when stochastic sources are present.

Again omitting details of the argument, the result is that in limit cycles of the deterministic part of the motions — i.e.  $\dot{y}_1 = \dot{y}_2 = \dot{y}_3 = 0$  and  $dw_1, dw_2, dw_3$  ignored in equations (4.9)*a, b, c* — the variable  $y_2$  can be expressed as a function of  $y_1$  only:

$$y_2(y_1) = \begin{cases} \frac{\alpha_2^* \beta}{(\alpha_2^*)^2 + (\theta^*)^2} & (y_1 < y_1^*) \\ y_2^* & (y_1 > y_1^*) \end{cases} \quad (4.13)$$

where

$$y_2^* = \frac{\alpha_2^* - \alpha_1^* + 0.5\delta_1}{3\beta} \quad (4.14)$$

$$\delta_1 = \frac{\pi}{4} \left( S_1(0) + S_1(2\omega_1) + \frac{1}{\omega_1^2} S_2(2\omega_1) \right) \quad (4.15)$$

and  $y_1^*$  is the smallest positive root of

$$y_1^2 \frac{\alpha_2^* \beta}{(\alpha_2^*)^2 + (\theta^*)^2} = y_2^* \quad (4.16)$$

Note that  $\alpha_2^*$ , equation (4.12)*b*, is a function of  $y_1$ .

Equation (4.13) is the equation of the center manifold, essentially a relation between the amplitudes in the limit cycle; substitution of (4.13) in (4.9)*c*, with  $\dot{y}_3 = 0$  and  $dw_3$  dropped gives the values of  $y_3$  in the center manifold.

### 4.4 The Fokker-Planck-Kolmogorov Equation for Equations (4.9)

We now confine attention to the behavior in the limit cycle executed in the presence of stochastic sources. Assuming that the variable  $y_2$  is sufficiently well-approximated by its dependence on the center manifold, equation (4.9)*a* is now a stochastic differential equation for  $y_1$  only:

$$dy_1 = \left( \alpha_1^* y_1 + \beta y_1 y_2(y_1) + \frac{\pi}{2y_1 \omega_1^2} S_3(\omega_1) \right) dt + \sqrt{Q_{11}} dw_1 \quad (4.17)$$

where  $y_2(y_1)$  is given by (4.13).

The amplitude  $y_1$  is now a random variable characterized by a probability density  $p(y_1, t)$ . With

known methods (e.g. Gardiner 1985, Chapter 5) we can construct the Fokker-Planck-Kolmogorov equation for  $p$ :

$$\frac{\partial p}{\partial t} = - \frac{\partial}{\partial y_1} \left[ \left( \alpha_1^* y_1 + \beta y_1 y_2(y_1) + \frac{\pi}{2y_1 \omega_1^2} S_3(\omega_1) \right) p \right] + \frac{1}{2} \frac{\partial^2}{\partial y_1^2} [Q_{11} p] \quad (4.18)$$

This equation possesses a stationary solution independent of time,

$$p^s(y_1) = C y_1 \exp \left\{ \int_0^{y_1} \frac{x(\delta_2 + 2\beta y_2(x))}{Q_{11}(x)} dx \right\} \quad (4.19)$$

where

$$\delta_2 = 2\alpha_1^* - 3\delta_1 \quad (4.20)$$

and  $C$  is a constant fixed by requiring that  $p^s(y_1)$  be normalized:

$$\int_0^\infty p^s(x) dx = 1 \quad (4.21)$$

The solution (4.19) is an approximation to the distribution of amplitude  $r_1$ , in the limit cycle when stochastic sources are assumed to affect directly only the fundamental mode. Just as for the deterministic behavior, the parameters arising in the problem must satisfy a stability condition in order that a stationary solution exist ("existence in mean square value"). Here, the deterministic conditions (4.7)*b* becomes

$$\alpha_2 + 2\alpha_1 + \pi \left[ \frac{3}{4} S_1(0) - \frac{1}{\omega_1^2} S_2(0) + \frac{3}{8} S_1(2\omega_1) + \frac{3}{8\omega_1^2} S_2(2\omega_1) \right] < 0 \quad (4.22)$$

Because the noise field is coupled to the acoustic field, producing both parametric and 'external' excitation according to the general equations (3.16), a reasonable question is: can the presence of stochastic sources be responsible for observed combustion instabilities? That is, are the noise sources sufficiently strong to cause excitation and sustenance of acoustic modes that are deterministically stable ( $\alpha_n < 0$ )?

To obtain meaningful results, we must ensure that a stationary probability distribution exists. The conditions to be satisfied are

$$\alpha_n + \frac{\pi}{4} \left[ S_1^{(n)}(2\omega_n) + \frac{1}{\omega_n^2} S_2^{(n)}(2\omega_n) \right] < 0 \quad (4.23)$$

where  $S_1^{(n)}$  and  $S_2^{(n)}$  are the power spectral density fluctuations of the parameter excitations  $\xi_n$  and  $\xi_n^v$  in the  $n^{\text{th}}$  acoustic mode. For technical reasons of analysis, we are ignoring the 'external' excitations  $\Xi_n$  which cannot cause modes to be excited under the conditions treated here.

#### 4.5 Some Numerical Results

Because we have not attempted to work out physically realistic models of the non-acoustic fluctuations ( $\tilde{u}$ ,  $\tilde{p}$ ,  $\tilde{\rho}$ ) we cannot claim or anticipate that the results faithfully represent observed behavior. The initial results discussed here indicate that the approach taken here, based on time and stochastic averaging, is an effective method for treating problems in which both instabilities and noise are present.

The analysis described above can be used to carry out three independent sorts of computations: Monte Carlo simulations with the second-order equations (4.1)a,b; Monte Carlo simulations with the first-order stochastic equations (4.9)a,b,c; and the probability distribution can be found directly as the solution to the Fokker-Planck-Kolmogorov equation (4.18).

Monte Carlo simulations with the second-order equations are very lengthy, requiring an estimated 100 CPU hours on a VAX 8800 for a simulation consisting of 100 samples, each covering 3,000 periods of the fundamental oscillation. Here we give some results of simulations with the first-order equations, requiring approximately 5 hours each. This work is continuing with other machines, but no results are available.

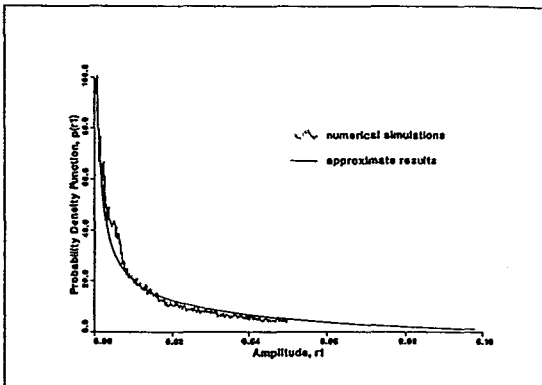


Figure 5.

Attention here is confined to conditions under which stationary probability distributions should exist, with linear parameters ( $\alpha_1 \dots$ ) assigned realistic values. Figure 5 is a comparison of the distribution given by equation (4.19) with a Monte Carlo simulation. The frequency of the fundamental mode is 800 hertz,  $\alpha_1 = 8 \text{ s}^{-1}$ , and  $\alpha_2 = -125 \text{ s}^{-1}$ ; both modes are damped. There is no external excitation ( $\Xi_1 = 0$ ) and only the parametric excitation  $\xi_1(t)$  is present. Good agreement is apparent. Figure 6 shows the time history of the amplitude of the first mode, computed for two samples chosen from the Monte Carlo simulations. Note that one is 'stable,' decaying within about 20 cycles, while the other exhibits 'asymptotic stability,' associated with the rather long tail of the distribution that is shown in Figure 1.

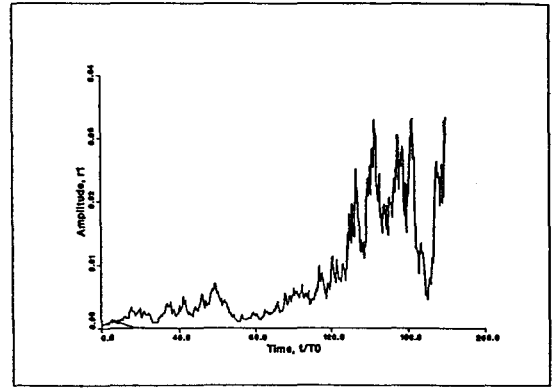


Figure 6.

Figure 7 is a comparison of the distribution computed with equation (4.19) with a Monte Carlo simulation when both parametric and external excitations are present ( $\xi_1(t)$ ,  $\xi_2(t)$ , and  $\Xi_1(t)$  all non-zero). The parameters  $\alpha_1$ ,  $\alpha_2$  have the same values used to produce Figure 5. Again the approximation (4.19) for the probability distribution is satisfactory. Note that the external excitation is capable of causing the fundamental modes to be excited to an amplitude somewhat larger than that occasionally caused by parametric excitation alone: the most probable and average values of  $r_1$  are larger than those predicted with the distribution given in Figure 1.

Figures 8 and 9 show cases in which the fundamental mode is unstable and the second is stable. The results therefore show the influence of stochastic sources on the amplitude of the first mode in a limit cycle. Now the most probable value of the amplitude naturally obtains a higher value, larger in Figure 9 for which  $\alpha_1$  has greater magnitude: in Figure 8,  $\alpha_1 = 8 \text{ s}^{-1}$  and in Figure 9,  $\alpha_1 = -25 \text{ s}^{-1}$ .

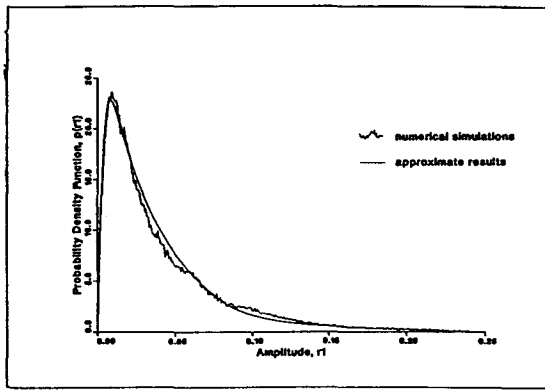


Figure 7.

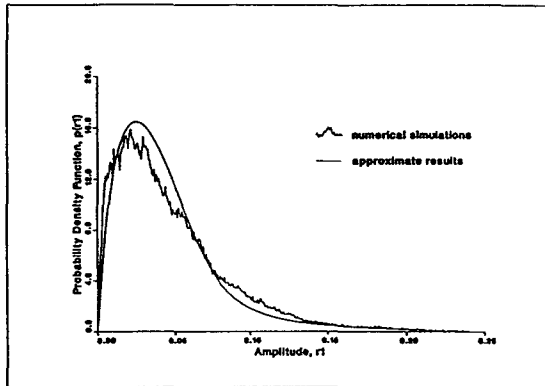


Figure 8.

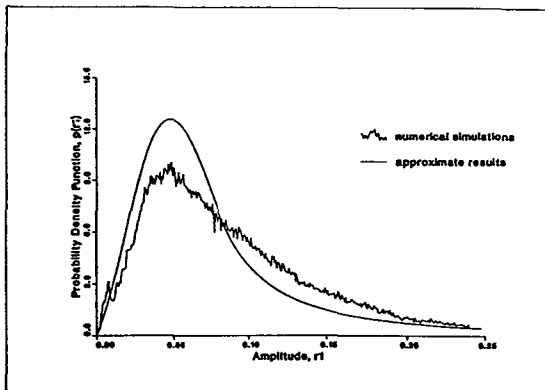


Figure 9.

## 5. Determining the Dimensions of Attractors Observed in Combustion Chambers

The noise field treated in the preceding section is produced by stochastic sources. Hence the unsteady motions generated and propagating in the chamber have the characteristics of random processes. In contrast, the motions identified as combustion instabilities constitute a deterministic

system. However, pressure records taken with operating systems often appear also to have a random character, raising the question of possible chaotic motions. It's an important question, not simply because this might be another example of deterministic chaos, but because the search for the answer should provide further information about unsteady nonlinear behavior in combustion chambers.

In the past decade much has been accomplished in developing methods of analyzing experimental data to find evidence of chaos. A central issue is determining the dimensions of attractors, objects in phase space to which the motions of a nonlinear system tend after long times, the subject of this section. A point in phase space uniquely defines the instantaneous state of a dynamical system. For the acoustic field defined by the system of equations (2.31), appropriate coordinates of phase space are the displacements and velocities ( $\eta_n, \dot{\eta}_n$ ) of the oscillators corresponding to the acoustic modes. In principle, the phase space may be infinite dimensional, but since we can treat only a finite number of modes (a minimum of two), the phase space will always have a finite number of dimensions, at least four.

A path or trajectory in phase space represents the evolution of motion of the dynamical system in time. An attractor is a point or a collection of points in phase space to which the motion tends after a long time. Thus if all modes are stable, the pressure fluctuations ultimately vanish and the origin ( $\eta_1 = \dot{\eta}_1 = \dots = \eta_n = \dot{\eta}_n = 0$ ) is an attractor, in this case having dimension 0. It is common in combustion systems exhibiting instabilities that the motions settle down to a limit cycle in which many modes may participate, but the motion is periodic. A limit cycle is represented by a closed (non-intersecting) path in phase space, so the attractor in this case has dimension one. Periodic limit cycles have been long known from theoretical work (e.g. Sirigano and Crocco 1964, Zinn 1968, Zinn and Powell 1971, Culick 1975, 1990, Culick and Yang 1989). It is important to note that in a limit cycle, the frequencies of all participating modes must be such that  $\omega_m/\omega_n$  is a rational number for all  $m, n$ . That means, as found, for example, in application of (2.31), that the frequencies of the linear acoustic modes for an arbitrary geometry must in general be shifted by nonlinear effects so that all ratios are indeed rational. For the results reported by Zinn and Powell (1971), Culick (1975, 1990), and Culick and Yang (1989), the frequencies in the limit cycle are integral multiples of the fundamental. That relation arose both for longitudinal modes, when  $\omega_n = n\omega_1$ , for the unperturbed (classical) modes,

and for modes of a cylindrical chamber when the classical frequencies are not in simple ratios.

If there are two or more fundamental frequencies in the limit cycle such that  $\omega_n^{(1)} = n\omega_1$  and  $\omega_m^{(2)} = m\omega_2$ , or if the frequencies of the participating modes form irrational ratios in ultimate steady motion, the corresponding attractor in phase space is a torus having integral dimension, 2, 3, 4, ... or greater, corresponding to 2, 3, 4, ... fundamental frequencies. Those two classes of possible motions are called periodic and quasi-periodic. Even though the phase space may have large dimension, just as limit cycles of dimension 1 may exist, so also may tori of, say, dimension 2 exist. It is possible, as a result of 'frequency-locking,' that a quasi-periodic motion may become periodic. Tori of dimension greater than 3 may be unstable (technically, 'structurally unstable'), and, if so, would not be observable. That is a matter of continuing research; consult Bergé, Pomeau, and Vidal (1984) for a good readable introduction to the subject.

It is now a familiar result that deterministic dynamical systems may exhibit apparently random behaviors called chaos, fundamentally different from true random or stochastic behavior, conveniently called simple noise. Although not theoretically necessary, it appears that for a real system to exhibit chaotic behavior, its phase space must contain an attractor having non-integral dimension, a strange attractor. The question addressed in this section is: can combustion chambers showing instabilities also develop chaotic motions?

To date it appears that no chaotic motions have been definitely identified in analytic or numerical results for the sorts of acoustical systems we are concerned with here, although some examples may have been found in results reported by Papanizos and Culick (1991). Recent work by Jahnke and Culick (1991) has suggested that toroidal attractors may exist, but the results are preliminary and the implications are unknown. It is therefore particularly important to investigate the possibility by examining experimental results. All recordings of pressure in combustion chambers show random fluctuations traditionally assumed to be noise. The problem, a common one, is to discover whether one can distinguish chaos in the noise.

The nonlinear acoustical system itself is of course not the only possible source of chaotic behavior in a combustion chamber. Indeed, much effort has been expended in the recent past on identifying chaotic behavior in many problems of fluid mechanics, with particular emphasis on the development of turbulence. Turbulent fluid flow is typically

characterized statistically even though it is governed by deterministic partial differential equations. The stochastic representation is useful because the number of degrees of freedom present in the fluid is very large (it can be approximated by dividing a characteristic volume by the cube of the Kolmogorov scale). However, in many systems, much simpler large-scale dynamics may emerge due to fluid mechanical feedback mechanisms that result in coherent oscillatory flows. The vortex dynamics of jets, wakes, and other separated flows are often modeled as having only a few degrees of freedom even though turbulence is present. Similarly, in combustion chambers, the presence of acoustic modes (given by equations 2.30a, b) often results in coherent oscillations that can be modeled effectively with only a few ordinary differential equations.

However, acoustic oscillations are present because some mechanism exists for transfer of energy from combustion processes or the flow itself (e.g. large-scale vortices) to the coherent motions. The behavior of the organized motions could therefore conceivably serve as a marker of chaotic behavior actually originating somewhere within the system. We are far from understanding the dynamics of the processes in combustion chambers theoretically, but in any event, progress with the theory depends heavily on experimental results. We report here some results obtained with analysis of pressure records taken in a laboratory dump combustor fueled with pre-mixed gases (Sterling and Zukoski 1987, 1991).

Recently developed data analysis techniques may be used to determine the minimum number of degrees of freedom responsible for the generation of a particular time-series signal. The techniques, developed as an outgrowth of modern nonlinear dynamical systems theory during the past decade, make use of a signal processing technique known as the 'time-delay embedding method.' This method allows an experimentalist to embed the data in a phase-space representation. If start-up transients have ceased, the phase-space trajectory traces out a set of points that comprise an attractor. The dimension of this attractor provides a lower limit on the number of degrees of freedom of the system active in the observed nonlinear behavior. Details on both the embedding method and the dimension-finding algorithms can be found in an introductory article by Gershenfeld (1988).

The method, proposed first by Packard *et al.* (1980), rests on the idea that some properties of a multi-dimensional nonlinear system may be determined from the time-series measurements of a single variable. In particular, under quite broad circumstances often prevailing in actual situations,

a reconstruction of the attractor according to the following procedure will have (almost always) the same dimension as the true attractor. In the present case, we have a recording of the pressure as a function of time, such as that reproduced in Figure 10. Choose a fixed time  $t$  when the pressure is  $p(t)$  and construct the  $n$ -dimensional vector  $P(t)$  having components equal to  $p(t)$  and the values of the pressure at the delayed times  $t + \tau, t + 2\tau, \dots, t + (n-1)\tau$ :

$$P(t) = \{p(t), p(t+\tau), p(t+2\tau), \dots, p(t+(n-1)\tau)\} \quad (5.1)$$

This representation contains two parameters: the delay time  $\tau$  and the embedding dimension  $n$ . Both must be varied with care as noted below.

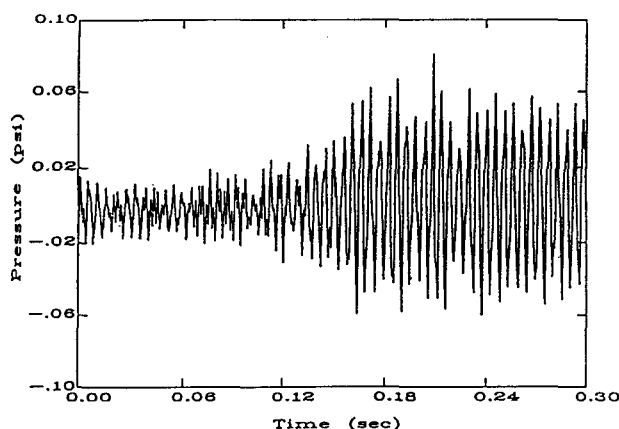


Figure 10.

After  $\tau$  and  $n$  have been chosen,  $t$  is assigned discrete values  $t_1, t_2, \dots$  (i.e. the pressure record is digitized); the vectors  $P(t_i)$  then trace a trajectory in the  $n$ -dimensional embedding space. That trajectory is the reconstruction of the attractor. Theoretical results have shown that if the attractor associated with the motion of the system has dimension  $D$ , then the dimension of its reconstruction will also be  $D$  if  $n > 2D+1$  (see, e.g., reviews by Parker and Chua 1987 and by Theiler 1990). The choice of  $\tau$  is rather delicate: it is sufficient here to remark that it must be neither too small nor too large or too close to the period of a motion in the system. Thus  $\tau$  is varied in the data processing to produce best results.

The problem now is to estimate the dimension of the object obtained with the procedure just described. Consider a point on the reconstructed attractor and imagine a ball or hypersphere of radius  $r$  centered on the point. The number of points within the ball depends on the size of the ball, varying as  $r^\nu$  where  $\nu$  is the dimension of the attractor (see Bergé, Pomeau, and Vidal 1984, p. 149 for a supporting

argument); thus if  $N(r)$  is the number of points on that portion of the attractor on the ball,

$$N(r) = Ar^\nu \quad (5.2)$$

where  $A$  is a constant and  $\nu$  is the dimension of the attractor. Hence

$$\nu = \frac{\log N(r) + \text{constant}}{\log r} \quad (5.3)$$

and a plot of  $\log N(r)$  versus  $\log r$  will have slope equal to the dimension.

That idea has been extended by Grassberger and Procaccia (1983) to define a 'correlation dimension.' Consider pairs of points located at the tips of pairs of vectors  $P(t_i), P(t_j)$ , defined by (5.1). The distance between a pair,  $|P(t_i) - P(t_j)|$ , can be computed and the spatial correlation  $C(r)$  is formed:

$$C(r) = \lim_{N \rightarrow \infty} \frac{1}{N^2} \left\{ \begin{array}{l} \text{number of pairs of points for} \\ \text{which } |P(t_i) - P(t_j)| < r \end{array} \right\} \quad (5.4)$$

where  $N$  is the total number of pairs considered. The correlation dimension  $\nu$  is defined in analogy with (5.2), writing

$$C(r) = Ar^\nu \quad (5.5)$$

Hence

$$\nu = \frac{\log C(r) + \text{constant}}{\log r} \quad (5.6)$$

If a plot of  $\log C(r)$  versus  $\log r$  shows a linear variation, the slope is identified as the dimension of the attractor.

The number of points  $P(t_i)$  used depends in the first instance on the length of the pressure record available and on the interval of digitizing, both of which can be varied, of course. We assume that a sufficiently large number is used, so the chief remaining parameter is the embedding dimension  $n$  which yields important information when it is varied. Suppose first that the attractor is a limit cycle, a curve of dimension one. If the chosen points  $P(t_i)$  are distributed uniformly (for example) on the curve, then for sufficiently small  $r$ , it seems reasonable to expect  $N(r) \sim r$  so  $\nu = 1$  as required. This result doesn't change if the embedding dimension is varied because the points must always lie on the curve.

The case is different if noise is present. As discussed in Section 2, we may view the noise field as the superposed motions of a large number of modes or oscillators (an infinite number for ideal white noise). Hence the trajectories of the system lie in a phase space of large (infinite) dimension and densely cover any embedding space of finite (small) dimension.

Hence the criterion  $n \geq 2d + 1$  cannot be satisfied for small  $n$ . As a result (Bergé, Pomeau, Vidal 1984, pp. 152) the dimension  $\nu$  computed according to (5.6) approaches the embedding dimension as  $r$  is decreased. In other words, if the procedure based on the idea of correlation dimension is applied to a stochastic or random process, a dimension  $\nu$  cannot be found independently of the embedding dimension.

In this work we used a version of the algorithm of Grassberger and Procaccia (1983) written by Schneider (1990). Figure 11 shows results obtained for a random signal (the 'data points' were computed with a random number generator). For the two choices of embedding dimension  $n = 5, 9$ , the slope of  $\log C(r)$  versus  $\log r$  does not approach a constant value as  $r$  (the size of the ball) is reduced. As expected by construction of the data, there is no evidence of a low-dimensional attractor, in confirmation of the remarks in the preceding paragraph.

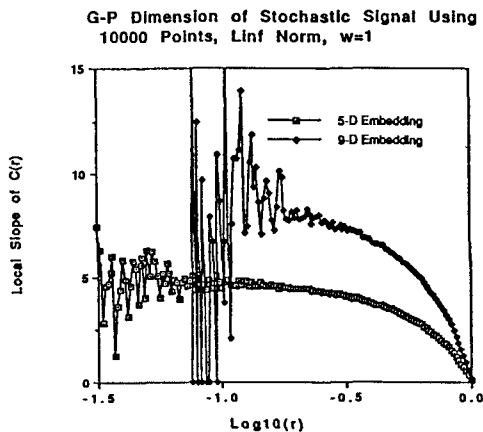


Figure 11.

Data from the combustor at the Caltech Jet Propulsion Center (Sterling and Zukoski 1987) has been obtained when a premixed methane/air flame was stabilized behind a rearward-facing step. The power was around 45 kW based on the higher heating value of the fuel. A pressure transducer located above the flameholder was used to monitor the oscillatory pressure and data was collected at 80 microsecond intervals. The frequency spectrum of the pressure signal is shown in Figure 12. The prominent peaks correspond to a longitudinal acoustic mode near 460 hertz, a mode near 530 hertz, and the 1/2 and 4/5 subharmonics of the latter mode that are present near 265 and 424 hertz.

Spectrum of Pressure Oscillations

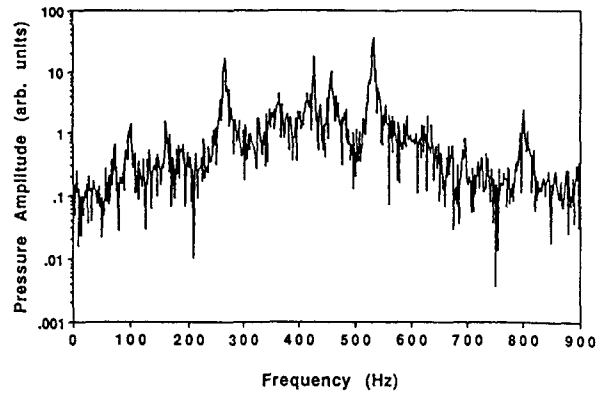


Figure 12.

Figure 13 shows the dimension of the signal that was used to generate the spectrum of Figure 12. The points on the attractor, which numbered 8150, were embedded in a 9-D phase space using a 400 microsecond (5 points) time delay. This particular case reveals significant scale separation between noise and deterministic dynamics. A two-dimensional attractor (a 2-torus) is observed over approximately an order of magnitude of scales. A value of two is a result of the excitation of the two acoustic modes. Since the other peaks in the spectrum are subharmonics of these fundamental modes, they are not associated with additional degrees of freedom but are a result of the nonlinear nature of the oscillations, as described in the remarks at the beginning of this section. This dimension determination provides a lower bound on the number of modes that participate in the large-amplitude oscillations. However, the dimension increases abruptly at the smaller scales, suggesting that additional degrees of freedom are present, either due to the presence of additional acoustic modes or as a result of the turbulent combustion.

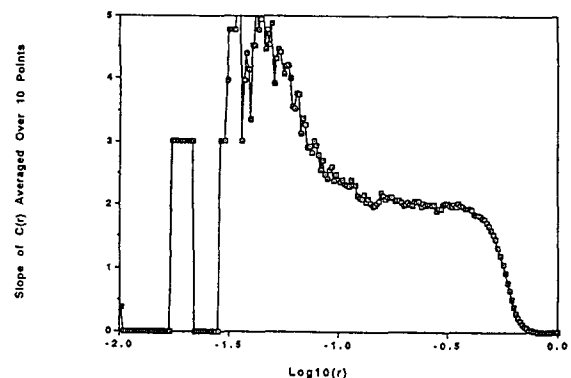


Figure 13.

With the limited results obtained to date we have found no evidence for an attractor having fractional dimension, and hence no suggestion of chaotic behavior. The two oscillations defining the toroidal attractor at 460 Hz and 530 Hz correspond to linear acoustic modes of the system (Sterling and Zukoski 1991), but the results of the data analysis do not establish the precise mechanism causing those modes to be excited, or the processes responsible for the existence of the attractor. Nor do we yet have any understanding of possible interactions between the attractor and the noise in the system. Observations reported in earlier publications have shown that the excitation and sustenance of the steady large-amplitude acoustical motions are associated with the shedding of large vortical structures from the rearward-facing step, combustion within the vortices, and interactions with the lateral boundaries of the duct. No analysis has been done for the complete problem.

We believe that the method used here (only one of several known approaches) to extract information about the nonlinear behavior from pressure records can provide important contributions to understanding unsteady motions in combustion chambers. Results obtained in this fashion cannot alone explain observations, or serve as a basis for possible means of active control. Meaningful progress will require collaboration with other methods of the sort described earlier in this paper.

## 6. Concluding Remarks

The approximate analysis discussed in the first part of this paper has been used successfully for many years to investigate problems of stabilities in combustion chambers. Extension to accommodate stochastic sources representing possible effects of combustion noise is new and only a few results have been obtained. Although the approach taken here seems to produce results consistent with observed behavior, we cannot draw definite conclusions.

In particular, we have not yet carried out sufficient calculations to determine the influences of deterministic motions on noise levels. Therefore, we cannot assess the proposal to control noise levels by controlling the acoustic field. It may not be possible to make such an assessment without incorporating realistic models of the noise sources. We have avoided that issue by assuming that the non-acoustic fluctuations are simply Gaussian noise. That cannot be an accurate assumption even though the sources will in any case enter the analysis as parametric and external excitations of the form appearing in the system equations (3.16). Modeling the non-acoustical motions will likely be the most difficult and important part of the problem: we

have no reason to change the general structure of the analytical framework.

Analysis of experimental data with the methods suggested in Section 5 clearly merits continued effort. The information obtained is likely to provide important contributions to understanding the nonlinear behavior. The same methods can of course be applied to numerical data obtained with the analysis described earlier. Thus one is able to work simultaneously with modeling physical processes, results of the approximate analysis, and experimental data, a necessary strategy for dealing with problems that cannot be solved entirely theoretically from first principles.

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## Discussion

QUESTION BY: R. Artaz, SNECMA, France

1. You are introducing a parallel between the wave equation of your model and the one of Lilley. As the latter holds in a free environment, how can you justify the comparison for an application to combustion noise in a confined environment?
2. To that effect, how do you treat the limited aspect conditions of the environment?

AUTHOR'S RESPONSE:

1. My purpose has not been to suggest that the same problems have been solved, but only to indicate that the basic assumptions are almost identical in the two subjects of combustion instabilities and aerodynamic noise. It is not an extremely important point nor does it have practical consequences (I think). I make it only to clarify the sort of unity that does exist.
2. If I understand your term correctly, you refer to the presence of the bounding walls and the exhaust nozzle. Their influences appear mainly in the forms chosen for the basis functions (mode shapes) used in the expansion of the pressure field, and partly in the boundary conditions applied to the solution of the problem accounting for perturbations of the classical acoustics for a chamber.

QUESTION BY: J.M. Seiner, NASA Langley, USA

Have you considered application of bi-spectral methods to determine non-linear relationships of your observed fundamental combustion modes?

AUTHOR'S RESPONSE:

We have not. Although bi-spectral methods might provide some interesting information, I do not believe that they will give the sort of results we have sought in this work. We shall investigate your suggestion.